

ON PATTERSON'S CONJECTURE: SUMS OF QUARTIC EXPONENTIAL SUMS

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ABSTRACT. We give more evidence for Patterson's conjecture on sums of exponential sums, by getting an asymptotic for a sum of quartic exponential sums over $\mathbb{Q}[i]$. Previously, the strongest evidence of Patterson's conjecture over a number field is the paper of Livné and Patterson [LP] on sums of cubic exponential sums over $\mathbb{Q}[\omega]$, $\omega^3 = 1$.

The key ideas in getting such an asymptotic are a Kuznetsov-like trace formula for metaplectic forms over a quartic cover of GL_2 , and an identity on exponential sums relating Kloosterman sums and quartic exponential sums. To synthesize the spectral theory and the exponential sum identity, there is need for a good amount of analytic number theory.

An unexpected aspect of the asymptotic of the sums of exponential sums is that there can be a secondary main term additional to the main term which is not predicted in Patterson's original paper [P].

1. INTRODUCTION

The Riemann Hypothesis, Goldbach's Conjecture, Twin Primes conjecture, and Fermat's Last Theorem are all examples of lucid statements in number theory. Perhaps not as well known, Patterson's conjecture, as posed by S.J. Patterson in [P], is just as clear in its presentation.

To understand, we will motivate Patterson's conjecture. Given a polynomial f with integral coefficients and a rational prime p , we can ask how large the exponential sum

$$\sum_{x(p)} \exp\left(\frac{2\pi i f(x)}{p}\right)$$

is as $p \rightarrow \infty$. From Weil's result on the Riemann Hypothesis for curves over a finite field, we can conclude that the sum is bounded by $O(\sqrt{p})$. If we ask the same question for prime power or integral modulus, the "square-root" bound continues to hold via other techniques, like stationary phase, that are much easier to apply than Weil's bound. We will call this "square-root" bound the Weil bound whether it is for a prime, prime power, or integral modulus.

Naturally, the next question to ask is whether or not it is possible to improve on Weil's bound on average. Specifically, fix a large $X > 0$ and investigate how large the following sum of sums is:

$$(1.1) \quad \sum_{p \leq X} \sum_{x(p)} \exp\left(\frac{2\pi i f(x)}{p}\right).$$

Just using Weil's bound here and bounding the p -sum trivially (using the prime number theorem) would get the bound $O\left(\frac{X^{\frac{3}{2}}}{\log X}\right)$ for the sum. So we ask, can this bound be improved by allowing the exponential sums to "cancel" with each other as they are summed over the primes less than X ? This seems to be very difficult when we consider these exponential sums

with an integral polynomial as comparable with Kloosterman sums:

$$S(A, B, p) := \sum_{x(p)} \exp\left(\frac{2\pi i(Ax + \frac{B}{x})}{p}\right),$$

which does not seem unreasonable as they have comparable Weil bounds and a further connection to be discussed later. Assuming this comparison, showing for a fixed $\delta > 0$, the sum

$$\sum_{p \leq X} S(0, 1, p) = \sum_{p \leq X} \mu(p) \ll X^{1-\delta}$$

implies a quasi-Riemann hypothesis. So this slight improvement over the trivial bound (which is $O(\frac{X}{\log X})$) would be a huge achievement in this analogous case. So we do not expect to make significant progress on improving the bound (1.1).

We can then try to improve the Weil bound as we sum over positive integers rather than over primes:

$$(1.2) \quad \sum_{c \leq X} \sum_{x(c)} \exp\left(\frac{2\pi i f(x)}{c}\right).$$

Again the bound to beat would be $O(X^{3/2})$ for (1.2). There is hope that the sum over primes, which we initially mentioned, can be still be understood via a sieve on the sum over integers. Two examples of using a sieve on a sum of exponential sums over integers include [HB] and [HBP] which study the distribution of the angle associated to a cubic Gauss sum of prime modulus. A third example is [DFI] which studies the distribution of roots of a quadratic polynomial with integral coefficients modulo a prime. The sums of exponential sums over primes arise in all three examples by translating the associated distribution problems using Weyl equidistribution. We should also say in all three cases improving on the Weil bound on average was crucial to the resolution to the distribution problems.

With the motivation now to improve upon Weil's bound on average, we state Patterson's conjecture.

Conjecture 1.1 (Patterson's Conjecture). *Let $f \in \mathbb{Z}[x]$, with $\deg f = n > 2$. If we let*

$$S(f, X) := \sum_{c \leq X} \sum_{x(c)} \exp\left(\frac{2\pi i f(x)}{c}\right)$$

then for a constant K depending on f ,

$$(1.3) \quad S(f, X) \sim \begin{cases} KX^{1+2/n} & \text{if } f(x) = f(a-x) \text{ for some } a \in \mathbb{Z}; \\ KX^{1+1/n} & \text{if else.} \end{cases}$$

We note that this conjecture immediately implies an improvement over using the Weil bound and trivially summing over the integral modulus. Patterson conjectures such an asymptotic in [P] with plenty of numerical experimentation and a couple examples which we now mention. In [P3], Patterson obtained the asymptotic for any $\epsilon > 0$,

$$S(Ax^3, X) = K_A X^{\frac{4}{3}} + O(X^{\frac{5}{4}+\epsilon}).$$

Though we restrict to f with $\deg f > 2$, we can also study $S(Ax^2 + Bx + C, X)$ with $B \neq 0$ by standard methods using quadratic Gauss sums and quadratic reciprocity to show for $\delta > 0$,

$$S(Ax^2 + Bx + C, X) = K_{A,B,C} X^{\frac{3}{2}} + O(X^{\frac{3}{2}-\delta}).$$

Both of these examples support Patterson's conjecture. However, this seems to be the extent of what we can prove about Patterson's conjecture over \mathbb{Z} , so we ask about an analog of Patterson's conjecture over the integers of a number field.

Take our number field to be $\mathbb{Q}[\omega]$, $\omega^3 = 1$. Let $e(\text{Tr}(z)) = \exp(2\pi i(z + \bar{z}))$, and define $S(f(x), X)$ accordingly. Livné and Patterson [LP] proved that

Theorem 1.2. *Suppose that $A, B, D, D' \in \mathbb{Z}[\omega]$ are such that D, D' are square-free and $(D, D') = 1$, both are co-prime to 3, that any prime $(\neq \sqrt{-3})$ which divides B also divides D' , that $27 \cdot A \mid B^3$, and that $\frac{B^3}{27A} \equiv \pm 1(3)$. Let χ be a Dirichlet character of modulus dividing D' . Then for any $\epsilon > 0$, we have*

$$(1.4) \quad \sum_{\substack{\mathbb{N}(c) \leq X, (c, D')=1 \\ c \equiv 0(D), c \equiv 1(3)}} S(Ax^3 + Bx, X) \chi(c) = \frac{(2\pi)^{1/3} \Gamma(1/3)^2 \sqrt{3}}{32 \zeta_{\mathbb{Q}[\omega]}(2)} \frac{\mathbb{N}(D')}{\sigma(DD')} X^{\frac{4}{3}} + O(X^{\frac{5}{4}+\epsilon})$$

if $\chi = \left(\frac{A}{\cdot}\right)_3$ the cubic residue symbol, and

$$\sum_{\substack{\mathbb{N}(c) \leq X, (c, D')=1 \\ c \equiv 0(D), c \equiv 1(3)}} S(Ax^3 + Bx, X) \chi(c) = O(X^{\frac{1}{4}+\epsilon}) \text{ otherwise. Here } \sigma(D) = \prod_{p \text{ prime}, p \mid D} (1 + \mathbb{N}(p)).$$

It is not clear how this theorem fits immediately into the above conjecture for sums of exponential sums over \mathbb{Z} , but it definitely is analogous. The incorporation of Dirichlet characters into the number field case generalizes Patterson's conjecture at the same time as solving a technical aspect of studying such sums. We will discuss more this technical need for the Dirichlet character in Section 2.

1.1. Main Theorem. We now state the main theorem of the paper, which considers exponential sums over the Gaussian integers $\mathbb{Z}[i]$. One should consider this number field as \mathbb{Q} adjoined the fourth roots of unity just as [LP] considered in Theorem 1.2, \mathbb{Q} adjoined the cube roots of unity. In this case our exponential sums are defined as $e(\text{Tr}(z)) = \exp(2\pi i(z + \bar{z}))$.

For $\psi \in C_0^\infty(\mathbb{R}^+)$ and for large $X > 0$, denote $\hat{\psi}(s)$ as the Mellin transform over \mathbb{R} , specifically

$$\hat{\psi}(s) = \int_0^\infty \psi(y) y^{s-1} dy.$$

Theorem 1.3. *Suppose $A, B, F, D \in \mathbb{Z}[i]$ are such that $D \equiv 1(4)$ is square-free, that $B = 4B'$ and that any prime p dividing B' also divides D , and that $\frac{B^2}{16A} \in \mathbb{Z}[i]$. Let θ be a Dirichlet character modulo D . Suppose also that A, B are squares. Let α be the best bound toward the Ramanujan conjecture for automorphic representations over $\mathbb{Q}[i]$ (the current best bound is $\alpha = \frac{7}{64}$). Then for any $\epsilon > \frac{\alpha}{4}$, there exist a constant K depending on A, B, D such that*

$$(1.5) \quad \sum_{\substack{c \in \mathbb{Z}[i], (c, D)=1 \\ c \equiv 1(4)}} \frac{\psi(\frac{X}{\mathbb{N}(c)})}{\mathbb{N}(c)} \theta(c) \sum_{x(c)} e(\text{Tr}(\frac{Ax^4 + Bx^2 + F}{c})) =$$

$$\begin{cases} X^{\frac{1}{4}} \hat{\psi}(-\frac{1}{4})K + O(X^\epsilon) & \text{if } \theta^4 \equiv \mathbf{1}(D), \theta \neq \mathbf{1}(D); \\ X^{\frac{1}{2}} \hat{\psi}(-\frac{1}{2})\mathbb{N}(A) + X^{\frac{1}{4}} \hat{\psi}(-\frac{1}{4})K + R + O(X^\epsilon) & \text{if } \theta \equiv \mathbf{1}(D); \\ O(X^\epsilon) & \text{if } \theta^4 \not\equiv \mathbf{1}(D), \end{cases}$$

with

$$R = \frac{L(1, \chi_{\mathbb{Q}[i]})\phi(4D)}{\mathbb{N}(4D)} \int_{(-1/2, 1/2)} X^{1/4+\delta/2} \left[\sum_{n \in \mathbb{Z}[i] - \{0\}} \frac{\psi(\frac{X^{1/2-\delta}}{\mathbb{N}(n)})}{\mathbb{N}(n)} \sum_{\substack{x(n)^* \\ x+\bar{x} \equiv -4(n)}} \left(\frac{x}{n}\right)_4 \right] d\delta$$

where $\left(\frac{x}{n}\right)_4$ is the quartic residue symbol.

Remarks. • The constant K will depend on Fourier coefficients of a metaplectic residual Eisenstein series over the quartic cover of GL_2 . We will make this constant very explicit in Section 7.1.

- If $\theta \equiv \mathbf{1}(D)$, then the third term of the asymptotic R we had difficulties dealing with. We fully expect the term is $o(X^{\frac{1}{4}})$ due to the oscillation of the term

$$\sum_{\substack{x(n)^* \\ x+\bar{x} \equiv -4(n)}} \left(\frac{x}{n}\right)_4.$$

Using a Weil bound one can obtain the bound $O(X^{\frac{1}{2}+\epsilon})$, but this does not help with Patterson's conjecture. If we replace the quartic residue character with a trivial character then, by completing the square, we are counting solutions to quadratic equations modulo n . Such a sum is seen in counting geodesics on the modular surface in the work of Young and Soundararajan [SY]. Another example of this sum arising is in a subconvexity estimate for the Rankin-Selberg L-function in the author's work [H].

- Another vantage point of introducing the character $\theta \bmod(D)$ is that we see a kind of “inner product” in Theorem 1.3 by averaging over c of θ against the sum quartic exponential sum. That the inner product is large if $\theta^4 \equiv \mathbf{1}(D)$, and small for any other θ would be a facet we would like to see in a more general Patterson's conjecture.

We consider Theorem 1.3 as a smoothed version of Patterson's conjecture. With more careful analysis, the smoothing could be removed with loss in the error term. We also normalized by a $\frac{1}{\mathbb{N}(c)}$ -factor to connect such sums more easily to the spectral theory of metaplectic forms which are crucial to the result. A summation by parts argument would remove this normalization if desired.

Recall from Patterson's conjecture that if $f(a-x) = f(x)$ then we expect different behavior for $S(f(x), X)$, namely it is asymptotic to $X^{1+\frac{2}{n}}$ instead of $X^{1+\frac{1}{n}}$. In our Theorem 1.3, $f(x) = Ax^4 + Bx^2 + F = f(-x)$, and so we do see in the case θ is the trivial character, that our main term is asymptotic to $X^{\frac{1}{2}} = X^{\frac{2}{4}}$, which is correct with our normalization of the c -sum. Believing that Patterson's conjecture would generalize to number fields, in that analogous sums would have analogous main terms in X , Livné and Patterson's Theorem 1.2 as well as our Theorem 1.3 are in agreement. What was not expected is that also in these two choices of θ , there is a secondary main term. It is not clear if such secondary terms exists from the data in [P] for exponential sums over \mathbb{Z} .

Another feature of our theorem is that if θ is the quartic residue character or a quadratic character, then the main term of size $X^{\frac{1}{2}}$ vanishes, and we are left with a main term of size $X^{\frac{1}{4}}$, which is what we would expect in Patterson's conjecture from a polynomial f having no $a \in \mathbb{Z}[i]$ such that $f(a-x) = f(x)$. This sort of property tells us that a correct version of Patterson's conjecture for number fields will be much more delicate to state than for \mathbb{Q} .

If θ is a character not of order 4, then our theorem looks much similar to Theorem 1.2 for the case $\chi^3 \neq 1$. Our error term is much better due to the smoothing as well as using the strongest results known towards the Ramanujan conjecture for automorphic forms, which then via the Shimura correspondence gives strong bounds for metaplectic forms.

2. OUTLINE OF PROOF

An important role in our theorem is played by sums of Kloosterman sums. In particular, asking for a similar asymptotic to Patterson's conjecture for a sum of Kloosterman sums. Such asymptotics have been studied using the spectral theory of automorphic forms originally by Kuznetsov [Kuz] and Goldfeld and Sarnak [GS]. This connection between sums of Kloosterman sums and automorphic forms is well known and is seen through the Kuznetsov or relative trace formula. In its most trivial form this connection can be seen for a $f \in C_0^\infty(\mathbb{R}^+)$ for large $X > 0$ as

$$(2.1) \quad \sum_{c=1}^{\infty} \frac{S(n, m, c)}{c} f\left(\frac{X}{c}\right) = \sum_{\substack{\pi \\ \frac{1}{2} < s_\pi < 1}} X^{s_\pi - \frac{1}{2}} \tilde{f}(s_\pi) a_\pi(n) \overline{a_\pi(m)} + O(X^\epsilon)$$

for any $\epsilon > 0$ and \tilde{f} some Bessel transform of f . The Kloosterman sum here is $S(n, m, c) = \sum_{x(c)^*} e\left(\frac{mx + n\bar{x}}{c}\right)$. The sum over π is of automorphic forms of level one with eigenvalue $s_\pi(1 - s_\pi)$. Forms with eigenvalues in this range $\frac{1}{2} < s_\pi < 1$ are called exceptional. In this special case of level one forms over \mathbb{Q} , there are no exceptional forms (see [DeshI]) and this sum of Kloosterman sums has no asymptotic but just a bound $O(X^\epsilon)$ for any $\epsilon > 0$. The non-existence of such exceptional forms for general congruence subgroups is the Selberg eigenvalue conjecture, which is intimately tied to the Ramanujan conjecture for automorphic representations.

Rather than look at automorphic forms over \mathbb{Q} we look at forms over $\mathbb{Q}[i]$. As well, we do not look quite at automorphic forms but at metaplectic forms on $\mathbb{Q}[i]$. These are functions on $\Gamma \backslash SL_2(\mathbb{C})/SU(2)$, for a certain discrete subgroup Γ , that—at their simplest—transform on the left by the quartic power residue symbol. We will be more explicit about their definition in Section 4. Also, analogous to automorphic forms, there is an associated spectral theory of metaplectic forms and a similar connection to sums of Kloosterman sums. However, in this case we know a metaplectic form with an exceptional parameter exists, namely the quartic theta function which is a residual Eisenstein series. This was discovered by Kubota in [Ku]. Label the residual Eisenstein series ψ_{00} with eigenvalue parameter s_{00} . Then the quartic metaplectic analog of (2.1) in its simplest form is

$$(2.2) \quad \sum_{c \in \mathbb{Z}[i], c \equiv 1(4)}^{\infty} \frac{S_4(n, m, c)}{\mathbb{N}(c)} f\left(\frac{X}{\mathbb{N}(c)}\right) = X^{s_{00}-1} \tilde{f}(s_{00}) a_{\psi_{00}}(n) \overline{a_{\psi_{00}}(m)} + \sum_{\substack{\pi \\ 0 < s_\pi < s_{00}}} X^{s_\pi-1} \tilde{f}(s_\pi) a_\pi(n) \overline{a_\pi(m)} + O(X^\epsilon).$$

Here we have

$$S_4(n, m, c) = \sum_{x(c)^*} \left(\frac{x}{c}\right)_4 e\left(\text{Tr}\left(\frac{mx + n\bar{x}}{c}\right)\right).$$

So how does this sum of metaplectic Kloosterman sums relate to a sum of quartic exponential sums in Theorem 1.3? As mentioned in the introduction, there is a connection or identities between Kloosterman sums and other exponential sums. The exponential sum for

a prime modulus in Theorem 1.3 (without the term $e(\frac{F}{p})$) can be written as

$$\sum_{x(p)} e(\text{Tr}(\frac{Ax^4 + Bx^2}{p})) = \sum_{x(p)} [(\frac{x}{p})_2 + 1] e(\text{Tr}(\frac{Ax^2 + Bx}{p})).$$

In Section 10, we prove

$$(2.3) \quad \sum_{x(p)} \left(\frac{x}{p}\right)_2 e(\text{Tr}(\frac{Ax^2 + Bx}{p})) = \left(\frac{AB}{p}\right)_2 e(\text{Tr}(\frac{-B^2\overline{8A}}{p})) \sum_{b(p)} \left(\frac{b}{p}\right)_4 e(\text{Tr}(\frac{\overline{4^2AB^2}(b + \overline{b})}{p})).$$

If we consider a sum over $c \in \mathbb{Z}[i]$ of the second sum $\sum_{x(c)} e(\text{Tr}(\frac{Ax^2+Bx}{c}))$, it is straightforward to understand its asymptotic. We study such a sum in Section 10. However, if we took a sum over $c \in \mathbb{Z}[i]$ of the first sum, then the asymptotic is not straightforward. In Section 8, we prove for $c \equiv 1(4)$, $(AB, c) = 1$,

$$(2.4) \quad \sum_{x(c)} e(\text{Tr}(\frac{Ax^4 + Bx^2}{c})) = \left(\frac{AB}{c}\right)_2 e(\text{Tr}(\frac{-B^2\overline{8A}}{c})) S_4(\overline{4^2AB^2}, \overline{4^2AB^2}, c) + \sum_{x(c)} e(\text{Tr}(\frac{Ax^2 + Bx}{c})) + \{\text{Cross terms}\}.$$

For simplicity of exposition, let us assume the asymptotic of the “cross” terms are negligible (though this is not quite true). If we can “swap out” the metaplectic Kloosterman sum using the above identity, then we will be able to get an asymptotic for something that looks very similar to a sum of quartic exponential sums. In essence, that is what is done in [LP] for sums of cubic exponential sums over $\mathbb{Q}[\omega]$. There the connection between Kloosterman sums and exponential sums is simpler and seen in the identity for $c \equiv 1(3)$,

$$(2.5) \quad \sum_{x(c)} e(\text{Tr}(\frac{Ax^3 + Bx}{c})) = (\frac{\overline{A}}{c})_3 \sum_{y(c)^*} (\frac{y}{c})_3 e(\text{Tr}(\frac{y - B^3\overline{3^3Ay}}{c})).$$

2.1. Archimedean analog of exponential sums. The cubic sum we just mentioned has an analogous identity from the archimedean perspective which is called Nicholson’s identity:

$$\int_0^\infty \cos(x^3 + Bx) dx = \frac{B^{1/3}}{3} K_{1/3}(2(y/3)^{3/2}).$$

The connection between the exponential sum and archimedean identity is more readily seen when rewritten (see [DI]) as

$$\int_{-\infty}^\infty \exp(i(Ax^3 + x)) dx = \frac{1}{\sqrt{3}} \int_0^\infty (x/A)^{1/3} \exp(-x - (3^3 Ax))^{-1} \frac{dx}{x}.$$

The archimedean analog of the quartic identity of (2.3) is

$$\int_0^\infty \cos(Ax^2) \sin(Bx) \frac{dx}{\sqrt{x}} = \frac{\sqrt{B}}{2\sqrt{A}} \cos(\frac{B^2}{8A} - \frac{3\pi}{8}) J_{1/4}(\frac{B^2}{8A}),$$

using ([GR], 6.686) and $J_{1/2}(Bx) = \sin(Bx) \sqrt{\frac{2}{\pi x}}$.

Our simplified explanation above, for understanding the asymptotic for certain sums of quartic exponential sums via a spectral theory of metaplectic forms, assumes virtually no technicalities. The problem is that there are many difficulties in the transition between the sums of Kloosterman sums and quartic exponential sums, and we explain those now.

2.2. Choice of trace formula. The first technical point is that we clearly need a trace formula for metaplectic forms. It turns out we can adapt a trace formula for automorphic forms over an imaginary quadratic field for our metaplectic spectrum. Let us mention the many trace formulas for automorphic forms we can choose from. In [BM], Bruggeman and Miatello create a Kuznetsov trace formula over any number field, but they choose the two Poincare series needed in the construction of the formula both at the cusp ∞ . In [BMo] and [L], Bruggeman and Motohashi and Lokvenec-Guleska, respectively, construct a trace formula over $\mathbb{Q}[i]$ with representations of all weights and expansions at different cusps, but the test functions involved—since they do include all representations of various weights—are difficult to analyze. We mention Louvel in [Lo] tried to do a similar analysis to what we need in this paper using the trace formula of [L]. However, his analysis of test functions on the spectral side of the trace formula is not strong enough for the results we need. This seems to be due to the extra representations of non-zero weight. In fact, since we are only concerned with the residual Eisenstein series of weight 0, having the extra representations of non-zero weight is unnecessary. Another formula is due to Bruggeman, Miatello, and Pacharoni [BMP1] where they construct a trace formula over a totally real field with different cusps; it is very similar to [BM] in design.

We chose to create our own trace formula in the appendix that is a direct consequence of combining the ideas of [BM] over an imaginary quadratic field and [BMP1] which uses various cusps. The advantage of this Bruggeman-Miatello Kuznetsov trace formula is that it is directly amenable to the estimates of another paper of Bruggeman-Miatello [BM1]. There the authors take their trace formula over a rank-one real group (which includes our case) and give asymptotics for sums of Kloosterman sums. The key to getting such estimates, which we describe in detail in Section 9, is that by choosing the test function f on the spectral side of the trace formula, we deal with a Bessel transform of f on the geometric side of the trace formula, and by some careful analysis in [BM1], we can realize this Bessel transform as a much simpler Mellin transform of f plus a negligible error term.

2.3. Choice of cusps. The reason we need a Kuznetsov trace formula with freedom to choose two different cusps is to synthesize the connection from Kloosterman sums to quartic exponential sums via the identity (2.4). We note from (2.4) we looked at the “unramified” case where $(AB, c) = 1$, but this does not take care of all cases. Livné and Patterson [LP] seeing the same problem, cleverly have all primes that divide A, B divide the level D of the metaplectic forms connected to the Kloosterman sum in question. Then looking at the trace formula at specific cusps, the c -sum will be co-prime to D , so “unramified” cases are all the cases and there are no cases where $(AB, c) > 1$. They call this a technical point that needs to be removed for a more general statement, but in both theirs and our case, it will require much deeper local analysis of these kind of exponential sum identities in (2.4) and (2.5) in “ramified” situations.

The choice of cusp will depend on the metaplectic forms we apply to the sum of quartic exponential sums having a nontrivial level D and nebentypus. The concern then is, does there exist a residual Eisenstein series over the quartic cover of GL_2 that transforms by not only the quartic residue symbol but also by this nebentypus character θ of modulus D ? The work of Kubota [Ku] or Patterson's work on the cubic cover does not cover these situations. So in Section 7 we show that such a residual Eisenstein series exists when D is square-free.

2.4. The “cross” terms. If we accept that the sums of Kloosterman sums and sums of quadratic exponential sums are understood we are left with the asymptotics of terms we call

“cross” terms which look like

$$\sum_{n,m \in \mathbb{Z}[i]} \psi\left(\frac{X}{\mathbb{N}(nm)}\right) \left[\sum_{x(m)} e\left(\frac{\overline{n}(Ax^2 + Bx)}{m}\right) \right] \left[e\left(\frac{-B^2 \overline{8Am}}{n}\right) S_4(-B^2 \overline{m4^2A}, n) \right].$$

To analyze this term we use Dirichlet characters. If we assume $\theta \neq \mathbf{1}(D)$, then the “cross” terms are negligible. However, for $\theta \equiv \mathbf{1}(D)$ we cannot discern what happens, though the term R in Theorem 1.3 seems similar to counting solutions to quadratic equations modulo n and summing over n in a dyadic interval.

2.5. Appendix. As we mentioned earlier the first part of the Appendix is concerned with a construction of a Kuznetsov trace formula over $\mathbb{Q}[i]$ with multiple cusps.

The second part of the appendix is more of a commentary on how far we can extend the exponential sum identities like (2.4) in connecting exponential sums with integral polynomial arguments and GL_2 Kloosterman sums. It seems likely that any higher degree polynomials will not be associated to degree 2 Kloosterman sums.

3. PRELIMINARIES

3.1. Notation. Let $G = SL_2(\mathbb{C})$.

As we are borrowing the trace formula from [BM] and [BMP1], our notation is completely analogous.

For $x \in \mathbb{C}$, we let $Tr(x) = 2\Re(x)$ and $\mathbb{N}(x) = x\bar{x}$ denote the standard trace and norm.

3.1.1. Subgroups of G . For $y \in \mathbb{R}^+$ we put

$$a[y] := \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}.$$

This is the identity component of a maximal \mathbb{R} -split torus in G . We normalize the Haar measure by $da = \frac{dy}{y}$.

For $x \in \mathbb{C}$, we let

$$n[x] := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in G.$$

The normalization of the Haar measure for $N := \{n[x] : x \in \mathbb{C}\}$ is $\frac{dx d\bar{x}}{-2\pi i}$.

For $u \in \mathbb{C}^*$ we define

$$b[u] := \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}.$$

Let M be the subgroup $\{b[u] : |u| = 1\}$ of $K = SU(2)$. We then have the Iwasawa decomposition $G = NAK$ with standard Parabolic subgroup NAM .

3.2. Cusps. Let Γ be a finite index subgroup of $SL_2(\mathbb{Z}[i])$, then it follows that Γ has a finite number of cusp classes. Let \mathcal{P} be a set of representatives of those classes. For each $\sigma \in \mathcal{P}$, we fix g_σ such that $\sigma = g_\sigma \cdot \infty$.

For each $\sigma \in \mathcal{P}$ there is a parabolic subgroup $P^\sigma := g_\sigma P g_\sigma^{-1}$, with decomposition $P^\sigma = N^\sigma A^\sigma M^\sigma$, with $N^\sigma := g_\sigma N g_\sigma^{-1}$, $A^\sigma = g_\sigma A g_\sigma^{-1}$, and $M^\sigma = g_\sigma M g_\sigma^{-1}$. We use conjugation by g_σ to transport Haar measures of N , A , and M to N^σ , A^σ , and M^σ .

For each cusp $\sigma \in \mathcal{P}$ and $g \in G$ we have unique decomposition $g = n_\sigma[g] g_\sigma a_\sigma[g] k_\sigma[g]$ with $n_\sigma[g] \in N^\sigma$, $a_\sigma[g] \in A$, and $k_\sigma[g] \in K$.

We choose $a^{-2\rho} dnda$ as the left invariant measure on G/K , where we note $a[y]^\rho = \sqrt{\mathbb{N}(y)}$.

3.3. Discrete subgroups. Let $\Gamma_{P^\sigma} := \Gamma \cap P^\sigma$ and $\Gamma_{N^\sigma} := \Gamma \cap N^\sigma$. There is a lattice t_σ such that $\Gamma_{N^\sigma} = \{g_\sigma n[\xi]g_\sigma^{-1} : \xi \in t_\sigma\}$.

3.4. Characters. We put $t'_\sigma = \{r \in \mathbb{Q}[i] : \text{Tr}(rx) \in \mathbb{Z} \text{ for all } x \in t_\sigma\}$. All characters of N^σ are $\chi_r : g_\sigma n[x]g_\sigma^{-1} \rightarrow e^{2\pi i \text{Tr}(rx)}$ for $r \in \mathbb{C}$. All characters of $\Gamma_{N^\sigma} \backslash N^\sigma$ are obtained for $r \in t'_\sigma$. We identify χ_r with the character $n[x] \rightarrow e^{2\pi i \text{Tr}(rx)}$. For a character θ of Γ , we say a cusp σ is *essentially cuspidal* if $\theta|_{\Gamma_{N^\sigma}} = 1$.

Of first importance for us are the subgroups of $\text{SL}_2(\mathbb{Z}[i])$ defined by

$$(3.1) \quad \Gamma_2 = \{\gamma \in \text{SL}_2(\mathbb{Z}[i]) : \exists g \in \text{SL}_2(\mathbb{Z}), \gamma \equiv g \pmod{4}\},$$

$$(3.2) \quad \Gamma_1(4) = \{\gamma \in \text{SL}_2(\mathbb{Z}[i]) : \gamma \equiv 1 \pmod{4}\},$$

$$(3.3) \quad \Gamma_0(D) = \{\gamma \in \text{SL}_2(\mathbb{Z}[i]) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{D}\}.$$

The Kubota symbol κ can now be introduced. It is defined on Γ_1 by

$$\kappa(\gamma) = \begin{cases} \left(\frac{a}{c}\right)_4 & \text{if } c \neq 0 \\ 1 & \text{if } c = 0, \end{cases} \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1,$$

with $\left(\frac{a}{c}\right)_4$ the quartic power residue symbol. This definition is then extended to Γ_2 by defining κ trivially on $\text{SL}_2(\mathbb{Z})$. More precisely, for any $\gamma_2 \in \Gamma_2$, there exists $g \in \text{SL}_2(\mathbb{Z})$ and $\gamma_1 \in \Gamma_1$ such that $\gamma_2 = g\gamma_1$, and we define

$$(3.4) \quad \kappa(\gamma_2) = \kappa(\gamma_1).$$

4. METAPLECTIC FORMS

Automorphic forms on 3-hyperbolic space $\mathbb{C} \times \mathbb{R}^+$ can be seen as automorphic forms on $\text{SL}_2(\mathbb{C})$ which are invariant by the action of the maximal compact subgroup K of $\text{SL}_2(\mathbb{C})$. In the rest of this section we shall consider the discrete subgroup of $\text{SL}_2(\mathbb{Z}[i])$, $\Gamma = \Gamma_1(4) \cap \Gamma_0(D)$ for D square-free. Let θ be a Dirichlet character modulo D over $\mathbb{Z}[i]$. We consider θ also as a character of $\Gamma_0(D)$ by

$$\theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \theta(d).$$

Let $L^2(\Gamma \backslash G, \theta\kappa)$ be the space of square-integrable functions on G which satisfy

$$f(\gamma g k) = \theta(\gamma)\kappa(\gamma)f(g) \quad \text{for all } \gamma \in \Gamma, k \in K.$$

Under the action of G by the regular representation, $L^2(\Gamma \backslash G, \theta\kappa)$ decomposes into irreducible unitary representations with finite multiplicities, say

$$(4.1) \quad L^2(\Gamma \backslash G, \theta\kappa) = \overline{\bigoplus V}.$$

Let us choose a complete orthonormal basis $\{f_l\}_{l \geq 0}$ of the space of K -finite vectors in the discrete spectrum, call it $L_d^2(\Gamma \backslash G, \theta\kappa)$, of $L^2(\Gamma \backslash G, \theta\kappa)$ such that each f_l generates one of the irreducible V under the action of G . The orthogonal complement of $L_d^2(\Gamma \backslash G, \theta\kappa)$ in $L^2(\Gamma \backslash G, \theta\kappa)$, call it $L_c^2(\Gamma \backslash G, \theta\kappa)$, is described by integrals of Eisenstein series.

As $G/K = \text{SL}_2(\mathbb{C})/\text{SU}_2(\mathbb{C})$, each $f_l \in L_d^2(\Gamma \backslash G, \theta\kappa)$, is an eigenvector of the Laplace operator L . We standardly normalize the Laplace operator to be $Lf_l = \lambda_l f_l$, with $\lambda_l =$

$1 - \mu_l^2, \mu_l \in i[0, \infty) \cup (0, 1]$. Forms with $\mu_l \in (0, 1]$ are said to have an *exceptional spectral parameter*.

4.1. Fourier Expansion of the discrete spectrum. Define

$$W_\sigma^{r,\nu}(ng_\sigma a[y]k) := \chi_r(n) \sqrt{N(y)} K_\nu(4\pi y \sqrt{N(r)}),$$

with K_ν the K -Bessel function of index ν . Let

$$d_{r,\sigma}(\nu) := \frac{1}{\text{vol}(\Gamma_{N\sigma} \setminus \Gamma)} \frac{2^{1-\nu} (2\pi \sqrt{N(r)})^{-\nu}}{\Gamma(\nu + 1)}.$$

Fix a cusp $g_\sigma \cdot \infty$ which is essentially cuspidal with respect to κ, θ . Then a metaplectic form f_l is invariant under $\Gamma_{N\sigma}$ and therefore has a Fourier expansion at the cusp $\sigma = g_\sigma \cdot \infty$. The Fourier expansion at the cusp σ is

$$f_l(g) = F_{0,\sigma} f_l(a_\sigma[g]) + \sum_{r \in t'_\sigma} c_{r,\sigma}(f_l) d_{r,\sigma}(\mu_f) W_\sigma^{r,\mu_l}(g),$$

with $F_{0,\sigma} f_l(a_\sigma[g])$ the Fourier term at $r = 0$. It is not crucial to explicate the zeroth Fourier coefficient, but if a form f_l has a vanishing zeroth Fourier coefficient for all cusps $g_\sigma \cdot \infty$ of $\Gamma \backslash G$, we call the form a *cuspidal form*.

4.2. Eisenstein Series. We now describe $L_c^2(\Gamma \backslash G, \theta\kappa)$. For each $\sigma \in \mathcal{P}$, there is an Eisenstein series

$$E(P^\sigma, \nu, i\mu, g) := \sum_{\gamma \in \Gamma_{P\sigma}/\Gamma} a_\sigma[\gamma g]^{\rho+2\nu\rho+i\mu}.$$

Here $\nu \in \mathbb{C}$, and μ is an element of a lattice in the hyperplane $\Re(x) = 0, x \in \mathbb{C}$. In particular, μ is defined by $a[\gamma]^\mu = 1$ for $\gamma \in \Gamma_{P\sigma}$, again using similar notation to [BM]. The series converges for $\nu > \frac{1}{2}$, and has meromorphic continuation in ν .

There is a Fourier expansion at a cusp σ' for $E(P^\sigma, \nu, i\mu, g)$ similar to the discrete spectrum. Namely, we have

$$(4.2) \quad E(P^\sigma, \nu, i\mu, na[z]k) = F_{0,\sigma,\sigma'}(\nu + i\mu, a[z]) + \sum_{r \in t'_\sigma} D_{r,\sigma}(P^\sigma, \nu, i\mu) d_{r,\sigma}(\nu + i\mu) W_{\nu+i\mu,r}(\sqrt{N(z)}),$$

with

$$F_{0,\sigma,\sigma'}(s, a[z]) = \frac{1}{\text{vol}(\Gamma_{N\sigma'} \setminus \Gamma)} \psi_{\sigma,\sigma'}(0, s) \frac{\pi}{s}$$

and

$$D_{r,\sigma}(P^\sigma, \nu, i\mu) = \frac{1}{\text{vol}(\Gamma_{N\sigma'} \setminus \Gamma)} \psi_{\sigma,\sigma'}(r, \nu + i\mu)$$

with

$$\psi_{\sigma,\sigma'}(r, s) = \sum_{c \in \sigma' \mathcal{C}^\sigma} \frac{1}{N(c)^s} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma' \mathcal{L}(c)^\sigma} \left(\theta \kappa \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^{-1} e(-Tr(\frac{rd}{c})).$$

Here $\sigma' \mathcal{C}^\sigma$ and $\sigma' \mathcal{L}(c)^\sigma$ are defined in the next section, specifically in Definition 5.1 and Proposition 5.2, respectively.

One feature of metaplectic forms that is not so readily seen in automorphic forms is that there exists a discrete non-cuspidal metaplectic form. According to Selberg's theory, such forms are residues of Eisenstein series. We will discuss this more in Section 7.

5. BRUGGEMAN-MIATELLO-KUZNETSOV TRACE FORMULA OVER AN IMAGINARY QUADRATIC FIELD WITH MULTIPLE CUSPS

In order to state the formula we need to define some terms. Let $s_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then the Bruhat decomposition for $SL_2(\mathbb{Q}[i])$ is a union of $SL_2(\mathbb{Q}[i]) \cup P$ and the big cell $C := (P \cup SL_2(\mathbb{Q}[i]))s_0(N \cup SL_2(\mathbb{Q}[i]))$.

Definition 5.1. Let $\sigma, \sigma' \in \mathcal{P}$. We define ${}^{\sigma'}\Gamma^\sigma := \Gamma \cup g_{\sigma'} C g_\sigma^{-1}$ and

$${}^{\sigma'}\mathcal{C}^\sigma := \{c \in \mathbb{Q}[i]^* : g_{\sigma'}^{-1} \gamma g_\sigma = \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix} \text{ for some } \gamma \in {}^{\sigma'}\Gamma^\sigma\}.$$

For each $c \in {}^{\sigma'}\mathcal{C}^\sigma$ we put ${}^{\sigma'}\Gamma(c)^\sigma := \{\gamma \in {}^{\sigma'}\Gamma^\sigma : g_{\sigma'}^{-1} \gamma g_\sigma = \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix}\}.$

Proposition 5.2. Let $\sigma, \sigma' \in \mathcal{P}$. For each $c \in {}^{\sigma'}\mathcal{C}^\sigma$ there is a finite set ${}^{\sigma'}\mathcal{L}(c)^\sigma$ such that

$${}^{\sigma'}\Gamma(c)^\sigma = \cup_{\gamma \in {}^{\sigma'}\mathcal{L}(c)^\sigma} \Gamma_{N\sigma'} \gamma \Gamma_{N\sigma}.$$

The set

$${}^{\sigma'}\mathcal{L}^\sigma := \cup_{c \in {}^{\sigma'}\mathcal{C}^\sigma} {}^{\sigma'}\mathcal{L}(c)^\sigma$$

is set of representatives for $\Gamma_{N\sigma'} \backslash {}^{\sigma'}\Gamma^\sigma / \Gamma_{N\sigma}$.

Proof. See [BMP1]. □

Definition 5.3. For $\psi > 0$ and $a > 6$ define $\mathcal{K}_a(\psi)$ as the set of even holomorphic functions k on $|\Re \nu| \leq 2\psi$ satisfying

$$k(\nu) \ll e^{-|\Im \nu|} (1 + |\Im \nu|)^{-a}.$$

Definition 5.4. Let ψ be a character of Γ . For $\sigma, \sigma' \in \mathcal{P}, r \in t'_\sigma, r' \in t'_{\sigma'}$, we define the Kloosterman sum

$$S_{\psi}^{\sigma, \sigma'}(r, r', c) := \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in g_{\sigma'}^{-1} {}^{\sigma'}\mathcal{L}(c)^\sigma g_\sigma} \left(\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^{-1} e(-\text{Tr}_{\mathbb{Q}(i)/\mathbb{Q}}(\frac{r'd}{c} + \frac{ra}{c})).$$

Definition 5.5. Let

$$\mathbf{B}(t, \nu) := \frac{I_\nu(4\pi\sqrt{t})I_\nu(4\pi\sqrt{\bar{t}}) - I_{-\nu}(4\pi\sqrt{t})I_{-\nu}(4\pi\sqrt{\bar{t}})}{\sin \pi \nu}.$$

For $k \in \mathcal{K}_a(\psi)$, we define the Bessel transform of k on \mathbb{C}^* by

$$Bk(t) := \frac{i}{2} \int_{\Re(\nu)=0} k(\nu) \mathbf{B}(t, \nu) \sin(\pi \nu) d\nu.$$

In this paper, we will always take the cusps σ, σ' to be non equal so there will be no “delta” term standardly seen in such Kuznetsov trace formulas. The most general form of the trace formula with possible equivalent cusps is in the appendix, Section 13.

For economy, let

$$CSC_{r, r'}^{\sigma, \sigma'}(k) := \sum_{\zeta \in \mathcal{P}} \sum_{\mu} \int_0^\infty \overline{D_{r, \sigma}(P\zeta, \frac{iy}{2}, i\mu)} D_{r', \sigma'}(P\zeta, \frac{iy}{2}, i\mu) d_{r, \sigma}(\mu) \overline{d_{r', \sigma'}(\mu)} k(\frac{iy}{2} + i\mu) dy,$$

where ζ is a sum over the cusps in \mathcal{P} .

Theorem 5.6. (*Bruggeman-Miatello-Kuznetsov Trace Formula*) Let $\sigma \neq \sigma' \in \mathcal{P}$, with character κ_θ of $\Gamma = \Gamma_1(4) \cap \Gamma_0(D)$ with D square-free. For $k \in \mathcal{K}_a(\psi)$, we have

$$(5.1) \quad \sum_{l \geq 1} \overline{c_{r,\sigma}(fl)} c_{r',\sigma'}(fl) d_{r,\sigma}(\mu_l) \overline{d_{r',\sigma'}(\mu_l)} k(\mu_l) + C S C_{r,r'}^{\sigma,\sigma'}(k) = \frac{\pi}{2} \sum_{c \in \sigma' \mathcal{C}^\sigma} S_{\kappa_\theta}^{\sigma,\sigma'}(r, r', c) |\mathbb{N}(c)|^{-1} B k\left(\frac{rr'}{c^2}\right).$$

6. THE TRACE FORMULA FOR EXPLICIT CUSPS

Once and for all, let us take for our essential cusps $g_{\sigma'} \cdot \infty = D - 1$ and $g_\sigma \cdot \infty$ with $g_{\sigma'} = \begin{pmatrix} 1-D & l \\ 1 & m \end{pmatrix}$ with $D, l \equiv 1(4), m \equiv 0(4)$ and $m \equiv 0(D)$ and $g_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We note they are not $\Gamma_0(D)$ -equivalent.

We now unravel the definitions of $\sigma' \mathcal{C}^\sigma$ and $\sigma' \mathcal{L}(c)^\sigma$ for our chosen cusps from the Kuznetsov trace formula above. Recall C is the big Bruhat cell and have elements of the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0, a, b, c, d \in \mathbb{Q}[i]$. Assume $g \in g_{\sigma'}^{-1} \sigma' \mathcal{L}(c)^\sigma$. We must have then $g_{\sigma'} g \in \Gamma_0(D) \cap \Gamma_1(4)$, or

$$\begin{pmatrix} (D-1)a + lc & (D-1)b + ld \\ -a + mc & -b + md \end{pmatrix} \in \Gamma_0(D) \cap \Gamma_1(4).$$

This implies $-a + mc \equiv 0(D)$ which implies $a \equiv 0(D)$, but as $ad - bc = 1$, it also follows $(c, D) = 1$. Also, this implies $a, d \equiv 0(4), c \equiv 1(4)$, and $b \equiv -1(4)$.

So it follows

$$\kappa(g_{\sigma'} \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \kappa(g_{\sigma'}) \left(\frac{a}{c}\right)_4 = \left(\frac{D}{1}\right)_4 \left(\frac{a}{c}\right)_4 = \left(\frac{a}{c}\right)_4.$$

As well

$$\theta(g_{\sigma'} \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \theta(-b + md) = \theta(-b),$$

but $ad - bc = 1$ and $a \equiv 0(D)$ implies $-b \equiv \bar{c}(D)$, so $\theta(-b) = \bar{\theta}(c)$.

We also recall that the map

$$(\gamma_{\sigma'}, \gamma_\sigma) \in \Gamma_{N^{\sigma'}} \times \Gamma_{N^\sigma} \rightarrow \gamma_{\sigma'} \gamma \gamma_\sigma \in \Gamma$$

is one-to-one and further the function $\gamma \rightarrow c(\gamma)$ is constant on each double class in $\Gamma_{N^{\sigma'}} \backslash \Gamma / \Gamma_{N^\sigma}$.

Recall $\sigma' \mathcal{L}(c)^\sigma$ is a set of representatives for $\Gamma_{N^{\sigma'}} \backslash \sigma' \Gamma(c)^\sigma / \Gamma_{N^\sigma}$. So the

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in g_{\sigma'}^{-1} \sigma' \mathcal{L}(c)^\sigma g_\sigma$$

in the definition of the Kloosterman sum is parametrized for a fixed c by $d \pmod{c}$.

We can then rewrite the trace formula in a more explicit form now.

Theorem 6.1. (*Bruggeman-Miatello-Kuznetsov Trace Formula (Variant)*)

Define cusps $g_{\sigma'} \cdot \infty = D - 1$ and $g_\sigma \cdot \infty$ with $g_{\sigma'} = \begin{pmatrix} 1-D & l \\ 1 & m \end{pmatrix}$ with $D, l \equiv 1(4), m \equiv 0(4)$ and $m \equiv 0(D)$ and $g_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let κ_θ be the character of $\Gamma = \Gamma_1(4) \cap \Gamma_0(D)$ with

D square-free. For $k \in \mathcal{K}_a(\psi)$, we have

$$(6.1) \quad \sum_{l \geq 1} \overline{c_{r,\sigma}(f_l)} c_{r',\sigma'}(f_l) d_{r,\sigma}(\mu_l) \overline{d_{r',\sigma'}(\mu_l)} k(\mu_l) + C S C_{r,r'}^{\sigma',\sigma}(k) =$$

$$\frac{\pi}{2} \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \theta(c) S_4(r, r', c) |\mathbb{N}(c)|^{-1} Bk\left(\frac{rr'}{c^2}\right)$$

with

$$S_4(r, r', c) := \sum_{ad \equiv 1(c)} \left(\frac{a}{c}\right)_4 e\left(\text{Tr}\left(\frac{r'd}{c} + \frac{ra}{c}\right)\right).$$

7. FINDING THE RESIDUAL EISENSTEIN SERIES

The next theorem is inspired by Theorem 3.1 from [LP].

Theorem 7.1. *Let $D \in \mathbb{Z}[i]$, $D \equiv 1(4)$, and square free.*

- (1) *Let $L_d^2(\Gamma \backslash G/K)$ denote the space of f_l satisfying*

$$f_l(\gamma g) = \theta(\gamma) \kappa(\gamma) f_l(g)$$

for a Dirichlet character $\theta \pmod{D}$, and $\gamma \in \Gamma_1 \cup \Gamma_0(D)$. Then there exists a $f_l \in L_d^2(\Gamma \backslash G/K)$ that is an eigenform for the Laplace operator L with spectral parameter $1 - \mu_l^2$ with $\mu_l = \frac{1}{4}$ only if θ^4 is trivial. Label this form with θ^4 trivial, f_{00} .

- (2) *Let α be the best bound toward the Ramanujan conjecture for automorphic representations over $\mathbb{Q}[i]$ (the current best bound is $\alpha = \frac{7}{64}$). Suppose that $f_l \neq f_{00}$ is an eigenform for the Laplace operator L with spectral parameter $1 - \mu_l^2$, then we have*

$$\Re(\mu_l) \leq \frac{\alpha}{4}.$$

Proof. Incorporating the above expressions for $\sigma' \mathcal{C}^\sigma$ and $\sigma' \mathcal{L}(c)^\sigma$ from Section 6, we have

$$(7.1) \quad \psi_{\sigma,\sigma'}(r, s) = \sum_{c \in \sigma' \mathcal{C}^\sigma} \frac{1}{\mathbb{N}(c)^s} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma' \mathcal{L}(c)^\sigma} \left(\theta \kappa \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^{-1} e\left(-\text{Tr}_{\mathbb{Q}(i)/\mathbb{Q}}\left(\frac{rd}{c}\right)\right) =$$

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{1}{\mathbb{N}(c)^s} \theta(c) \sum_{a(c)^*} \overline{\left(\frac{a}{c}\right)_4} e\left(-\text{Tr}_{\mathbb{Q}(i)/\mathbb{Q}}\left(\frac{r\bar{a}}{c}\right)\right).$$

Note the eigenvalue of the Laplace operator of $E(P^\sigma, s, 0, n \begin{pmatrix} \sqrt{z} & 0 \\ 0 & \frac{1}{\sqrt{z}} \end{pmatrix} k)$ is $(2-s)s$. If

we normalize $s = 1 + i\mu$, then the eigenvalue is $1 - \mu^2$ as in the statement of the proposition. So the goal is to show that the Eisenstein series has a pole in its constant term at $s = \frac{5}{4}$, if θ^4 is trivial. This implies the existence of a Residual Eisenstein series. This Residual Eisenstein series would have $\mu = \frac{1}{4}$ completing the proposition.

For $r = 0$, by an orthogonality of characters argument, $\psi_{\sigma,\sigma'}(0, s)$ is non-zero precisely when $c = c_0^4$, for $c_0 \in \mathbb{Z}[i]$. This gives

$$\psi_{\sigma,\sigma'}(0, s) = \sum_{\substack{c_0 \in \mathbb{Z}[i] \\ c_0 \equiv 1(4)}} \frac{\theta^4(c_0)\phi(c_0^4)}{\mathbb{N}(c_0)^{4s}}.$$

We use characters modulo 4 and that $\phi(c_0^4) = \mathbb{N}(c_0)^3\phi(c_0)$ to write the sum as

$$\psi_{\sigma,\sigma'}(0, s) = \frac{1}{2} \sum_{\lambda(4)} \sum_{c_0 \in \mathbb{Z}[i]} \frac{\phi(c_0)\theta^4(c_0)\lambda(c_0)}{\mathbb{N}(c_0)^{4s-3}}.$$

Now take the term with trivial $\lambda = 1$, the term in the sum in c_0 is invariant under transformation by a unit so can be written in terms of integral ideals

$$\frac{1}{2} \sum_{\epsilon \in \mathbb{Z}[i]^*} \sum_{(c_0)} \frac{\phi(c_0)\theta^4(c_0)}{\mathbb{N}(c_0)^{4s-3}} = 2 \frac{L(4s-4, \theta^4)}{L(4s-3, \theta^4)}$$

Clearly, the ratio of Hecke L-functions only has a simple pole at $s = \frac{5}{4}$ when θ^4 is trivial. Otherwise, there is no residual Eisenstein series.

We now prove the second statement using the Shimura correspondence [F] and the best bound towards the Ramanujan conjecture for a imaginary quadratic field [N]. The latter being an offspring of the result over \mathbb{Q} from [KSa]. Specifically, the Shimura correspondence tells us that the metaplectic forms are in one-to-one correspondence with certain automorphic forms. A metaplectic form on the quartic cover of GL_2 with spectral parameter $1 - \mu_l^2$ will correspond via the Shimura correspondence to an automorphic form of spectral parameter $1 - (4\mu)^2$. This comes directly from the Lemma of section 2.1 in [F]. The result of [N] gives that if there exists an exceptional spectral parameter μ for an automorphic form, then it must satisfy $\Re(\mu) \leq \frac{7}{64}$. So by the Shimura correspondence, the metaplectic form associated to this automorphic form with exceptional spectral parameter must then satisfy $4\Re(\mu) \leq \frac{7}{64}$, proving the second statement. \square

We also now can state our main theorem explicitly defining K from Theorem 1.3.

7.1. The explicit Theorem 1.3. Consider metaplectic forms in the space $L_d^2(\Gamma \backslash G/K, \theta\kappa)$. We see from Theorem 7.1 there exists a residual Eisenstein series for this space if $\theta^4 = 1$. So we again denote our theta function as f_{00} , with corresponding Fourier coefficients as above.

Denote $\hat{\psi}(s)$ as the Mellin transform over \mathbb{R} , specifically

$$\hat{\psi}(s) = \int_0^\infty \psi(y)y^{s-1}dy.$$

Then we have

Theorem 1.3 (Explicit) *Suppose $A, B, F, D \in \mathbb{Z}[i]$ are such that $D \equiv 1(4)$ is square-free, that $B = 4B'$ and that any prime p dividing B' also divides D , that $\frac{B^2}{16A} \in \mathbb{Z}[i]$. Suppose also that A, B are squares. Let θ be a Dirichlet character modulo D . Let α be the best bound toward the Ramanujan conjecture for automorphic representations over $\mathbb{Q}[i]$ (the current best bound is $\alpha = \frac{7}{64}$). Then for any $\epsilon > \frac{\alpha}{4}$,*

$$(7.2) \quad \sum_{\substack{c \in \mathbb{Z}[i], (c, D)=1 \\ c \equiv 1(4)}} \frac{\psi(\frac{X}{\mathbb{N}(c)})}{\mathbb{N}(c)} \theta(c) \sum_{x(c)} e(\text{Tr}(\frac{Ax^4 + Bx^2 + F}{c})) =$$

$$\begin{cases} X^{\frac{1}{4}} \hat{\psi}(-\frac{1}{4})K + O(X^\epsilon) & \text{if } \theta^4 \equiv \mathbf{1}(D), \theta \not\equiv \mathbf{1}(D); \\ X^{\frac{1}{2}} \hat{\psi}(-\frac{1}{2})\mathbb{N}(A) + X^{\frac{1}{4}} \hat{\psi}(-\frac{1}{4})K + R + O(X^\epsilon) & \text{if } \theta \equiv \mathbf{1}(D); \\ O(X^\epsilon) & \text{if } \theta^4 \not\equiv \mathbf{1}(D). \end{cases}$$

with

$$K = \frac{-\Gamma(3/4)^2}{\pi^2(8\pi)^{1/4}\Gamma(5/4)^2} \frac{|c_{\frac{B^2}{16A}, \sigma}(f_{00})|^2}{\text{vol}(\Gamma_{N^\sigma}/\Gamma)}$$

and

$$R = \frac{L(1, \chi_{\mathbb{Q}[i]})\phi(4D)}{\mathbb{N}(4D)} \int_{(-1/2, 1/2)} X^{1/4+\delta/2} \left[\sum_{n \in \mathbb{Z}[i] - \{0\}} \frac{\psi(\frac{X^{1/2-\delta}}{\mathbb{N}(n)})}{\mathbb{N}(n)} \sum_{\substack{x(n)^* \\ x+\bar{x} \equiv -4(n)}} \left(\frac{x}{c}\right)_4 \right] d\delta.$$

8. FROM KLOOSTERMAN SUMS TO QUARTIC EXPONENTIAL SUMS

In this section, though we work still over $\mathbb{Q}[i]$, for economy, we write $e(\text{Tr}x) = e(x)$.

Theorem 8.1. *For $p \equiv 1(4)$, η the quadratic character, ρ the quartic residue character, and $(AB, p) = 1$ we have*

$$(8.1) \quad \sum_{x(p)} e\left(\frac{Ax^4 + Bx^2}{p}\right) - \sum_{x(p)} e\left(\frac{Ax^2 + Bx}{p}\right) = \sum_{x(p)} \eta(x) e\left(\frac{Ax^2 + Bx}{p}\right)$$

with

$$\sum_{x(p)} \eta(x) e\left(\frac{Ax^2 + Bx}{p}\right) = \eta(AB) e\left(\frac{-B^2 \overline{8A}}{p}\right) \sum_{b(p)} \rho(b) e\left(\frac{\overline{4^2 AB^2}(b + \bar{b})}{p}\right).$$

Proof. First we investigate the multiplicative Fourier transform of the exponential sum in question,

$$(8.2) \quad \sum_{A(p)} \overline{\chi(A)} \left[\sum_{x(p)} \eta(x) e\left(\frac{Ax^2 + Bx}{p}\right) \right].$$

Assume $\chi = \mathbf{1}(p)$, by a change of variables $A \rightarrow A\overline{x^2}$, the sum in x is zero as $(B, p) = 1$. Now assume $\chi \neq \mathbf{1}$. By a change of variables $A \rightarrow A\overline{x^2}$, $x \rightarrow \overline{B}x$ this equals

$$\chi(\overline{B^2}) \eta(B) \tau(\chi^2 \eta) \tau(\overline{\chi}).$$

So by Fourier inversion

$$(8.3) \quad \sum_{x(p)} \eta(x) e\left(\frac{Ax^2 + Bx}{p}\right) = \eta(B) \frac{1}{\phi(p)} \sum_{\chi(p)} \chi(\overline{B^2}) \tau(\chi^2 \eta) \tau(\overline{\chi}) \chi(A).$$

The Hasse-Davenport relation gives the identity for $\chi(p)$,

$$(8.4) \quad \tau(\chi^n) = \frac{-\chi(n^n) \prod_{l|(n)} \tau(\chi\gamma^l)}{\prod_{l|(n)} \tau(\gamma^l)},$$

where γ is the n -th residue character.

Writing the term $\tau(\chi^2\eta) = \tau((\chi\rho)^2) = \tau((\chi\bar{\rho})^2)$, and using the Hasse-Davenport relation we can write

$$\tau((\chi\rho)^2) = \frac{\chi(4)\tau(\chi\bar{\rho})\tau(\chi\rho)\tau(\eta)\eta(-1)}{\mathbb{N}(p)}.$$

This equality can be reached by using the equalities $\eta\rho = \bar{\rho}$ and $\overline{\tau(\eta)} = \eta(-1)\tau(\eta)$.

So (8.3) equals

$$\frac{\eta(B)\tau(\eta)\eta(-1)}{\mathbb{N}(p)\phi(p)} \sum_{\chi(p)} \chi(\overline{B^2})\chi(A) [\chi(4)\tau(\chi\bar{\rho})\tau(\chi\rho)\tau(\bar{\chi})].$$

Opening all the Gauss sums and rearranging sums this equals

$$\frac{\eta(B)\tau(\eta)\eta(-1)}{\mathbb{N}(p)\phi(p)} \sum_{a(p)} \overline{\rho(a)}e\left(\frac{a}{p}\right) \sum_{b(p)} \rho(b)e\left(\frac{b}{p}\right) \sum_{c(p)} e\left(\frac{c}{p}\right) \sum_{\chi(p)} \chi(4abc\overline{B^2}A).$$

Using orthogonality of characters this equals

$$\frac{\eta(B)\tau(\eta)\eta(-1)}{\mathbb{N}(p)} \sum_{\substack{a,b,c(p) \\ 4abA \equiv B^2c(p)}} \overline{\rho(a)}e\left(\frac{a}{p}\right)\rho(b)e\left(\frac{b}{p}\right)e\left(\frac{c}{p}\right).$$

We can rewrite this as

$$\frac{\eta(B)\tau(\eta)\eta(-1)}{\mathbb{N}(p)} \sum_{a,b(p)} \rho(a)\overline{\rho(b)}e\left(\frac{a+b+4ab\overline{B^2}A}{p}\right).$$

A change of variables $b \rightarrow ba$ gives

$$(8.5) \quad \frac{\eta(B)\tau(\eta)\eta(-1)}{\mathbb{N}(p)} \sum_{b(p)} \overline{\rho(b)} \sum_{a(p)} e\left(\frac{4bA\overline{B^2}a^2 + (b+1)a}{p}\right).$$

We now need the standard result for $(c, 2a) = 1$, that

$$\sum_{y(c)} e\left(\frac{ay^2 + by}{c}\right) = \epsilon_c \sqrt{\mathbb{N}(c)} \eta(a) e\left(\frac{-4\overline{a}b^2}{c}\right).$$

with $\epsilon_c = \begin{cases} 1 & \text{if } c \equiv 1(4) \\ i & \text{if } c \equiv -1(4). \end{cases}$

Using this identity for (8.5) we get

$$(8.6) \quad \frac{\epsilon_p \sqrt{\mathbb{N}(p)} \eta(AB) \tau(\eta) \eta(-1)}{\mathbb{N}(p)} \sum_{b(p)} \eta(b) \overline{\rho(b)} e\left(\frac{-4^2 \overline{b} A B^2 (b+1)^2}{p}\right) = \\ \frac{\epsilon_p \sqrt{\mathbb{N}(p)} \eta(AB) \tau(\eta) \eta(-1)}{\mathbb{N}(p)} e\left(\frac{-B^2 \overline{8} A}{p}\right) \sum_{b(p)} \rho(b) e\left(\frac{4^2 \overline{A} B^2 (b + \overline{b})}{p}\right).$$

As $\tau(\eta) = \epsilon_p \sqrt{\mathbb{N}(p)}$ and $\epsilon_p^2 \eta(-1) = 1$, we finally have

$$\sum_{x(p)} \eta(x) e\left(\frac{Ax^2 + Bx}{p}\right) = \eta(AB) e\left(\frac{-B^2 \overline{8A}}{p}\right) \sum_{b(p)} \rho(b) e\left(\frac{\overline{4^2 AB^2}(b + \bar{b})}{p}\right).$$

□

8.1. Prime powers. We need the following proposition.

Proposition 8.2. *For $m > 1$ and $p \equiv 1(4)$ prime,*

$$(8.7) \quad S_\rho(a, b, p^{2m}) = \mathbb{N}(p)^m \sum_{\substack{u(p^{2m}) \\ u^2 \equiv b\bar{a}(p^{2m})}} \rho(u) e\left(\frac{2au}{p^{2m}}\right).$$

$$(8.8) \quad S_\rho(a, b, p^{2m+1}) = \mathbb{N}(p)^{m+1/2} \left(\frac{a}{p}\right)_2 \sum_{\substack{u(p^{2m+1}) \\ u^2 \equiv b\bar{a}(p^{2m+1})}} \rho(u) e\left(\frac{2au}{p^{2m+1}}\right)$$

Proof. See [P1].

□

Proposition 8.3. *Theorem 8.1 is true for prime powers if $B = 4B'$ with A, B are squares. Namely, for $m > 1$*

$$(8.9) \quad \sum_{x(p^m)} e\left(\frac{Ax^4 + Bx^2}{p^m}\right) - \sum_{x(p^m)} e\left(\frac{Ax^2 + Bx}{p^m}\right) = \eta(AB) e\left(\frac{-B^2 \overline{8A}}{p^m}\right) \sum_{b(p^m)} \rho(b) e\left(\frac{\overline{4^2 AB^2}(b + \bar{b})}{p^m}\right).$$

Proof. From Proposition 8.2 for odd powers, which would be considered the harder of the two cases, we have for $m \geq 1$,

$$(8.10) \quad \eta(AB) e\left(\frac{-B^2 \overline{8A}}{p^{2m+1}}\right) \sum_{b(p^{2m+1})} \rho(b) e\left(\frac{\overline{4^2 AB^2}(b + \bar{b})}{p^{2m+1}}\right) = \\ \eta(AB) e\left(\frac{-B^2 \overline{8A}}{p^{2m+1}}\right) \left[\mathbb{N}(p)^{m+1/2} \eta(A) \left(e\left(\frac{B^2 \overline{8A}}{p^{2m+1}}\right) + e\left(\frac{-B^2 \overline{8A}}{p^{2m+1}}\right) \right) \right] = \\ \eta(B) \mathbb{N}(p)^{m+1/2} \left[1 + e\left(\frac{-B^2 \overline{4A}}{p^{2m+1}}\right) \right].$$

Now we can write

$$(8.11) \quad \sum_{\substack{x(p^{2m+1}) \\ (x,p) > 1}} e\left(\frac{Ax^4 + Bx^2}{p^{2m+1}}\right) + \sum_{x(p^{2m+1})} \eta(x) e\left(\frac{Ax^2 + Bx}{p^{2m+1}}\right) = \sum_{x(p^{2m+1})} e\left(\frac{Ax^4 + Bx^2}{p^{2m+1}}\right) - \sum_{x(p^{2m+1})} \mathbf{1}(x) e\left(\frac{Ax^2 + Bx}{p^{2m+1}}\right).$$

We use Proposition 5.2 from [LP]. This proposition is an application of stationary phase. We will use the proposition on the first term on the above right hand side equation, the second term on the right hand side will follow analogously.

Following their notation, we have $f(x) = Ax^4 + Bx^2$, $f'(x) = 4Ax^3 + 2Bx$, and so the roots are $\alpha = 0$ and $\alpha^2 = \overline{2AB}$. Such a non-zero α exists as $\left(\frac{2}{p}\right)_2 = \left(\frac{2}{\mathbb{N}(p)}\right)_{2, \mathbb{Q}} = 1$ and $\left(\frac{-1}{p}\right)_2 = 1$

as $p \equiv 1(4)$ implies $\mathbb{N}(p) \equiv 1(8)$, and A, B are squares. Here $(\frac{a}{b})_{2, \mathbb{Q}}$ is the standard Legendre symbol. It is easy to check that $f''(\frac{-B}{2A}) = -4B$, so

$$\sum_{x(p^{2m+1})} e\left(\frac{Ax^4 + Bx^2}{p^{2m+1}}\right) = \mathbb{N}(p)^{m+1/2} \eta(B) \left[1 + 2e\left(\frac{-B^2 \overline{4A}}{p^{2m+1}}\right)\right].$$

Similarly,

$$\sum_{x(p^{2m+1})} \mathbf{1}(x) e\left(\frac{Ax^2 + Bx}{p^{2m+1}}\right) = \mathbb{N}(p)^{m+1/2} \eta(A) e\left(\frac{-B^2 \overline{4A}}{p^{2m+1}}\right).$$

That the right hand side of (8.11) is actually

$$\sum_{x(p^{2m+1})} e\left(\frac{Ax^4 + Bx^2}{p^{2m+1}}\right) - \sum_{x(p^{2m+1})} e\left(\frac{Ax^2 + Bx}{p^{2m+1}}\right),$$

comes from the easy to check fact that

$$\sum_{\substack{x(p^m) \\ (x,p) > 1}} e\left(\frac{Ax^2 + Bx}{p^m}\right) = 0$$

for m odd or even.

Again as A, B are perfect squares, the left hand side of (8.10) equals the left hand side of (8.11).

The even powers are completely analogous. □

8.2. Identity for composite numbers.

Proposition 8.4. *For $c \equiv 1(4)$, η the quadratic character, ρ the quartic residue character, and $(AB, c) = 1$ with A, B squares then we have*

$$(8.12) \quad \sum_{x(c)} e\left(\frac{Ax^4 + Bx^2}{c}\right) = \eta(AB) e\left(\frac{-B^2 \overline{8A}}{c}\right) \sum_{b(c)} \rho(b) e\left(\frac{\overline{4^2 AB^2(b + \bar{b})}}{c}\right) + \sum_{x(c)} e\left(\frac{Ax^2 + Bx}{c}\right) + \{\text{Cross terms}\}.$$

Here the “cross terms” are composed of the following terms: for any decomposition $c = nm$ with $(n, m) = 1, n \neq c, m \neq c$,

$$\left[\sum_{x(n)} e\left(\frac{\overline{m}(Ax^2 + Bx)}{n}\right) \right] \left[\eta_m(AB) e\left(\frac{-B^2 \overline{8An}}{m}\right) S_4(-B^2 \overline{n4^2 A}, m) \right].$$

Proof. It suffices to do this for $c = mn, (m, n) = 1$ with $m = p^k$ and $n = q^j$, p, q primes in $\mathbb{Z}[i]$. Clearly induction on the number of primes in the decomposition will complete the proposition.

Note by Chinese remainder theorem

$$(8.13) \quad \sum_{x(c)} e\left(\frac{Ax^4 + Bx^2}{c}\right) = \left[\sum_{x(n)} e\left(\frac{\overline{m}(Ax^4 + Bx^2)}{n}\right) \right] \left[\sum_{x(m)} e\left(\frac{\overline{n}(Ax^4 + Bx^2)}{m}\right) \right].$$

To show dependence of the quadratic character on its modulus we denote the quadratic character modulo p as $\eta_p(x)$.

We can write the right hand side of (8.13) using Theorem 8.1 and Proposition 8.3 as

$$(8.14) \quad \left[\sum_{x(n)} e\left(\frac{\overline{m}(Ax^2 + Bx)}{n}\right) + \eta_n(AB)e\left(\frac{-B^2\overline{8Am}}{n}\right)S_4(-B^2\overline{m8A}, n) \right] \times \\ \left[\sum_{x(m)} e\left(\frac{\overline{n}(Ax^2 + Bx)}{m}\right) + \eta_m(AB)e\left(\frac{-B^2\overline{8An}}{m}\right)S_4(-B^2\overline{n8A}, m) \right].$$

If we multiply out (8.14), the “end terms” are natural and can be written as

$$\sum_{x(nm)} e\left(\frac{Ax^2 + Bx}{mn}\right) = \sum_{x(c)} e\left(\frac{Ax^2 + Bx}{c}\right)$$

plus

$$\eta(AB)e\left(\frac{-B^2\overline{8A}}{nm}\right)S_4(-B^2\overline{8A}, nm) = \eta(AB)e\left(\frac{-B^2\overline{8A}}{c}\right)S_4(-B^2\overline{8A}, c),$$

with the first equality coming directly from Chinese remainder theorem and the second coming from twisted multiplicativity of Kloosterman sums:

$$\sum_{b(m)} \rho_m(b)e\left(\frac{\overline{4^2An}B^2(b + \bar{b})}{m}\right) \sum_{b(n)} \rho_n(b)e\left(\frac{\overline{4^2Am}B^2(b + \bar{b})}{n}\right) = \sum_{b(mn)} \rho_{mn}(b)e\left(\frac{\overline{4^2AB^2}(b + \bar{b})}{mn}\right).$$

The “cross” terms of multiplying out (8.14) are:

$$(8.15) \quad \left[\sum_{x(n)} e\left(\frac{\overline{m}(Ax^2 + Bx)}{n}\right) \right] \left[\eta_m(AB)e\left(\frac{-B^2\overline{8An}}{m}\right)S_4(-B^2\overline{n4^2A}, m) \right] + \\ \left[\sum_{x(m)} e\left(\frac{\overline{n}(Ax^2 + Bx)}{m}\right) \right] \left[\eta_n(AB)e\left(\frac{-B^2\overline{8Am}}{n}\right)S_4(-B^2\overline{m4^2A}, n) \right]$$

The point now is if $c = \prod_{j=1}^N p_j^{k_j}$, by multiplicativity of the individual Kloosterman sums and quadratic exponential sums, we can always reduce to a product of a single Kloosterman sum term times a single quadratic exponential sum. So again considering the case $c = nm$ with $(n, m) = 1$ is sufficient. \square

9. FROM THE TRACE FORMULA TO ASYMPTOTICS OF SUMS OF EXPONENTIAL SUMS

We use the trace formula from Section 13 which is also stated in Theorem 5.6. The point of this section is to use a clever choice of test function in this trace formula to get an asymptotic for a sum of Kloosterman sums. Thankfully, the hard work has been done for us in [BM1]. We summarize the key points of the argument.

To exploit the test functions in [BM1], we write from our above trace formula the Bessel transform as

(9.1)

$$Bk\left(\frac{4\pi mn}{c}\right) = \int_{\Re(\nu)=0} k(\nu) (I_{-\nu}(4\pi\sqrt{\frac{4\pi mn}{c}}) I_{-\nu}(4\pi\sqrt{\frac{4\pi mn}{c}}) - I_{\nu}(4\pi\sqrt{\frac{4\pi mn}{c}}) I_{\nu}(4\pi\sqrt{\frac{4\pi mn}{c}})) d\nu =$$

$$\int_{\Re(\nu)=0} k(\nu) g(\nu) \tau(\chi_m, \chi_n, \begin{pmatrix} \frac{1}{|c|} & 0 \\ 0 & |c| \end{pmatrix}, \nu) d\nu$$

with

$$g(\nu) = \frac{4(4\pi^2 |m_1 m_2|)^{\nu} \nu}{\Gamma(\nu + 1)^2},$$

and

$$\tau(\chi_m, \chi_n, \begin{pmatrix} \frac{1}{|c|} & 0 \\ 0 & |c| \end{pmatrix}, \nu) = \sum_{j,k \geq 0} \frac{(4\pi^2 \frac{mn}{c})^j (4\pi^2 \frac{mn}{c})^k}{j! k! \Gamma(\nu + 1 + j) \Gamma(\nu + 1 + k)}.$$

Here again $\chi_m\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = e(\text{Tr}(mx))$. The important point of writing the Bessel transform in this way is that we can use estimates from [BM1]. We will describe these estimates in the next section.

9.1. Asymptotics for sums of Kloosterman sums. Our goal in this section is to prove the following theorem.

Theorem 9.1. *Let $\psi \in C_0^\infty(\mathbb{R}^+)$, with transform $\tilde{\psi}(s) = \int_0^\infty \psi(\frac{1}{t}) t^{s-1} dt$, then for X large and for any $0 < \epsilon < 1/2$, we have*

$$\sum_{c \in \sigma' C^\sigma} \frac{S_{\theta\kappa}^{\sigma, \sigma'}(r, r', c)}{\mathbb{N}(c)} \psi\left(\frac{X}{\mathbb{N}(c)}\right) =$$

$$\sum_{\substack{l \geq 1 \\ \mu_l \in (0, 1]}} \overline{c_{r, \sigma}(f_l) c_{r', \sigma'}(f_l) d_{r, \sigma}(\mu_l) \overline{d_{r', \sigma'}(\mu_l)}} \frac{X^{\mu_l} \tilde{\psi}(\mu_l)}{g(-\mu_l)} + O(1 + X^{\epsilon-1+\log X}).$$

Proof. Choose $\psi \in C_0^\infty(\mathbb{R}^+)$ that is equal to 0 in neighborhood of 0 and 1 in a neighborhood of one. We let $\psi_X(y) := \psi(\frac{X}{y})$ with $X > 0$ large. So ψ_X has roughly support on $[X(1 - \delta), X(2 - \delta)]$, for $\delta > 0$ depending on the construction of ψ . Denote the Mellin transform of a function f as

$$\tilde{f}(\nu) = \int_0^\infty f(x) x^{s-1} dx.$$

We define the test function associated to our trace formula as

$$h_X(\nu) = \left[\frac{\tilde{\psi}_X(-\nu) - \tilde{\beta}_X(-\nu)}{g(\nu)} + \frac{\tilde{\psi}_X(\nu) - \tilde{\beta}_X(\nu)}{g(-\nu)} \right]$$

with β_X defined in [BM1]. We remind that with this definition, $h_X(\nu)$ is an even function in ν and holomorphic on the strip $|\Re \nu| \leq \xi$. For $|\Im \nu| \geq 1$, $|\Re \nu| \leq \xi$, we have

$$h_X(\nu) = O(X^{|\Re \nu|} e^{-|\Im \nu|} (1 + |\Im \nu|)^{-l}),$$

for any $l > 0$ that is sufficiently large. We also have the estimate for $|\Im \nu| \leq 1$, but $\pm \nu$ staying away from the zeroes of g ,

$$h_X(\nu) = O(X^{|\Re \nu|}).$$

Near the zeroes of g we have the estimate

$$h_X(\nu) = O(X^{|\Re \nu|}(1 + |\log X|)).$$

Since the zero of g is at $\nu = -1$, the size of this estimate is negligible. In particular $h_X \in K_a(\psi)$ for $l > a$ and $\xi > 2\psi$, and can be used in the trace formula.

If $\nu \in (0, 1]$ and is not a zero of g (which will be the case in this paper), the automorphic form associated to this spectral parameter is in the complementary series, and

$$(9.2) \quad h_X(\nu) = \frac{X^\nu \tilde{\psi}(\nu)}{g(\nu)} + O(X^{-\nu}).$$

Remark. In the paper [BM1], the variable $X \rightarrow 0$ while in our case we have $X \rightarrow \infty$. This is not a problem due to our definition of $h_X(\nu)$. Specifically, if $X \rightarrow 0$ then for $\nu \in \mathbb{R}$,

$$\frac{\tilde{\psi}(-\nu)X^{-\nu}}{g(\nu)}$$

will be the main term. This is their estimate while if $X \rightarrow \infty$ then the main term is

$$\frac{\tilde{\psi}(\nu)X^\nu}{g(-\nu)}.$$

Thus inverting the estimates of [BM1] so that $X \rightarrow \infty$ is straightforward.

We now state a crucial estimate from [[BM1], (8)]

$$\tau(\chi_m, \chi_n, \begin{pmatrix} \frac{1}{|c|} & 0 \\ 0 & |c| \end{pmatrix}, \nu) = 1 + O\left(\frac{|c|^{-2}}{1 + |\Im \nu|}\right)$$

on $-\epsilon \leq \Re \nu \leq \xi$ for each $\epsilon \in (0, 1/2)$ with the implicit constant depending on ϵ and ξ .

We note

$$Bh_X\left(\frac{rr'}{c^2}\right) = \int_{\Re(\nu)=0} h_X(\nu)g(\nu)\tau(\chi_r, \chi_{r'}, \begin{pmatrix} \frac{1}{|c|} & 0 \\ 0 & |c| \end{pmatrix}, \nu)d\nu$$

is in the notation of [BM1],

$$\widetilde{h}_X(ma) = \widetilde{h}_X\left(\begin{pmatrix} \frac{1}{|c|} & 0 \\ 0 & |c| \end{pmatrix}\right).$$

Directly following the analysis of [[BM1], 3.3] we have

$$(9.3) \quad Bh_X\left(\frac{rr'}{c^2}\right) = \frac{1}{\mathbb{N}(c)}\psi\left(\frac{X}{\mathbb{N}(c)}\right) + \frac{1}{\mathbb{N}(c)^{1+\epsilon}}O(X^{\epsilon-1} + X^{-1}),$$

for any $\epsilon > 0$.

Define

$$F_\psi(X) := \sum_{c \in \sigma' \mathcal{C}^\sigma} \frac{S_{\theta_K}^{\sigma, \sigma'}(r, r', c)}{\mathbb{N}(c)} \psi\left(\frac{X}{\mathbb{N}(c)}\right).$$

Then combining the Bruggeman-Miatello trace formula and the estimate (9.3) we have

$$(9.4) \quad \sum_{l \geq 1} \overline{c_{r, \sigma}(f_l)} c_{r', \sigma'}(f_l) d_{r, \sigma}(\mu_l) \overline{d_{r', \sigma'}(\mu_l)} h_X(\mu_l) + C S C_{r, r'}^{\sigma, \sigma'}(h_X) -$$

$$\frac{i}{2} \text{vol}(\Gamma_{N^\sigma} \setminus N^\sigma) \alpha(\sigma, r, \sigma', r') \int_{\Re(\nu)=0} k(\nu) \sin \pi \nu d\nu =$$

$$F_\psi(X) + O(X^{\epsilon-1}),$$

where the bound $O(X^{-1/2})$ is taken into the bound $O(X^{\epsilon-1})$ since $\epsilon > 0$ can be chosen much smaller than $1/2$. As we will always take nonequivalent cusps, the delta term is zero. For equivalent cusps, the estimate of this term is in [BM1].

Likewise, we break the spectral terms into those coming from residual Eisenstein series and those that are not. The latter, which is $\mathcal{S} \setminus (0, 1] = i[0, \infty)$, is estimated in [[BM1],(20)]

$$\sum_{\substack{l \geq 1 \\ \mu_l \in i[0, \infty)}} \overline{c_{r,\sigma}(f_l)} c_{r',\sigma'}(f_l) d_{r,\sigma}(\mu_l) \overline{d_{r',\sigma'}(\mu_l)} h_X(\mu_l) + C S C_{r,r'}(h_X) \ll O(1 + \log X),$$

(letting $Y = 4$ from [BM1].)

The other spectral terms, are only finite in number and using Theorem 1 from [BM1] we have

$$(9.5) \quad \sum_{\substack{l \geq 1 \\ \mu_l \in (0, 1]}} \overline{c_{r,\sigma}(f_l)} c_{r',\sigma'}(f_l) d_{r,\sigma}(\mu_l) \overline{d_{r',\sigma'}(\mu_l)} h_X(\mu_l) = \\ \sum_{\substack{l \geq 1 \\ \mu_l \in (0, 1]}} \overline{c_{r,\sigma}(f_l)} c_{r',\sigma'}(f_l) d_{r,\sigma}(\mu_l) \overline{d_{r',\sigma'}(\mu_l)} \frac{X^{\mu_l} \tilde{\psi}(\mu_l)}{g(-\mu_l)} + O(1 + X^{\xi-1}).$$

Using (9.4) and (9.5) the estimation

$$(9.6) \quad \sum_{c \in \sigma' C^\sigma} \frac{S_{\theta_\kappa}^{\sigma, \sigma'}(r, r', c)}{\mathbb{N}(c)} \psi\left(\frac{X}{\mathbb{N}(c)}\right) = \\ \sum_{\substack{l \geq 1 \\ \mu_l \in (0, 1]}} \overline{c_{r,\sigma}(f_l)} c_{r',\sigma'}(f_l) d_{r,\sigma}(\mu_l) \overline{d_{r',\sigma'}(\mu_l)} \frac{X^{\mu_l} \tilde{\psi}(\mu_l)}{g(-\mu_l)} + O(1 + X^{\epsilon-1} + \log X)$$

then follows for any $0 < \epsilon < 1/2$. This concludes Theorem 9.1. \square

10. QUADRATIC EXPONENTIAL SUMS

In this section we write $e(Tr(x)) = e(x)$. Suppose $A, B \in \mathbb{Z}[i]$ and $(AB, c) = 1$, then we recall the classical result

$$\sum_{y(c)} e\left(\frac{Ay^2 + By}{c}\right) = \epsilon_c \sqrt{\mathbb{N}(c)} \left(\frac{A}{c}\right)_2 e\left(\frac{-4\overline{A}B^2}{c}\right).$$

with $\epsilon_c = \begin{cases} 1 & \text{if } c \equiv 1(4) \\ i & \text{if } c \equiv -1(4). \end{cases}$

Let $\psi \in C_0^\infty(\mathbb{R}^+)$ with $\psi_X(y) := \psi(\frac{X}{y})$ for $X > 0$ large. For $D, F \in \mathbb{Z}[i]$ and D square-free, $2 \nmid D$ with $A \parallel D^\infty$, we study the asymptotic sum

$$(10.1) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c, D) = 1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) \sum_{x(c)} e\left(\frac{Ax^2 + Bx + F}{c}\right).$$

Using the above closed form expression for $\sum_{x(c)} e(\frac{Ax^2+Bx}{c})$, (10.1) equals

$$(10.2) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4)}} \frac{\psi_X(\mathbb{N}(c))e(\frac{F}{c})}{\sqrt{\mathbb{N}(c)}} \theta(c) \left(\frac{A}{c}\right)_2 e\left(\frac{-4\bar{A}B^2}{c}\right).$$

Using elementary reciprocity,

$$e\left(\frac{\bar{A}}{B}\right) = e\left(\frac{-\bar{B}}{A}\right) e\left(\frac{1}{AB}\right)$$

and quadratic reciprocity, this equals

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4)}} \frac{\psi_X(\mathbb{N}(c))e(\frac{B^2}{4Ac})e(\frac{F}{c})}{\sqrt{\mathbb{N}(c)}} \theta(c) \left(\frac{c}{A}\right)_2 e\left(\frac{\bar{c}B^2}{4A}\right).$$

The restriction on the c -sum we remove by Dirichlet characters modulo 4,

$$(10.3) \quad \frac{1}{4} \sum_{\lambda(4)} \sum_{c \in \mathbb{Z}[i]} \frac{\psi_X(\mathbb{N}(c))e(\frac{B^2}{4Ac})e(\frac{F}{c})}{\sqrt{\mathbb{N}(c)}} \lambda(c) \theta(c) \left(\frac{c}{A}\right)_2 e\left(\frac{\bar{c}B^2}{4A}\right).$$

We break the c -sum into arithmetic progressions modulo $Q := LCM(4, A, D)$ to get

$$(10.4) \quad \frac{1}{4} \sum_{\lambda(4)} \left[\sum_{d(Q)} \lambda(d) \theta(d) \left(\frac{d}{A}\right)_2 e\left(\frac{\bar{d}B^2}{4A}\right) \right] \sum_{c \equiv d(Q)} \frac{\psi_X(\mathbb{N}(c))e(\frac{B^2}{4Ac})e(\frac{F}{c})}{\sqrt{\mathbb{N}(c)}}.$$

Using Poisson summation on the k -sum in $c = d + Qk$, the interior sum can be written as

$$\sum_{c \equiv d(Q)} \frac{\psi_X(\mathbb{N}(c))e(\frac{B^2}{4Ac})}{\sqrt{\mathbb{N}(c)}} = \sum_{m \in \mathbb{Z}[i]} e\left(\frac{-dm}{Q}\right) \int_{\mathbb{C}} \psi\left(\frac{X}{\mathbb{N}(t)}\right) e\left(\frac{B^2}{4At}\right) e\left(\frac{F}{t}\right) e\left(\frac{-mt}{Q}\right) \frac{dt}{\sqrt{\mathbb{N}(t)}}.$$

In the integral, change variables $t \rightarrow \sqrt{X}t$ to get

$$(10.5) \quad \sqrt{X} \int_{\mathbb{C}} \psi\left(\frac{1}{\mathbb{N}(t)}\right) e\left(\frac{B^2}{4A\sqrt{X}t}\right) e\left(\frac{F}{\sqrt{X}t}\right) e\left(\frac{-m\sqrt{X}t}{Q}\right) \frac{dt}{\sqrt{\mathbb{N}(t)}}.$$

A standard integration by parts argument implies with $m \neq 0$ and large X (10.5) is $O(X^{-M})$ for any $M > 0$.

So (10.4) equals for any $M > 0$,

$$(10.6) \quad \frac{\sqrt{X}}{4} \sum_{\lambda(4)} \left[\sum_{d(Q)} \lambda(d) \theta(d) \left(\frac{d}{A}\right)_2 e\left(\frac{\bar{d}B^2}{4A}\right) \right] \int_{\mathbb{C}} \psi\left(\frac{1}{\mathbb{N}(t)}\right) e\left(\frac{B^2}{4A\sqrt{X}t}\right) e\left(\frac{F}{\sqrt{X}t}\right) \frac{dt}{\sqrt{\mathbb{N}(t)}} + O(X^{-M}).$$

To make any further calculations we assume the following: $B = 4B'$ and $B' \parallel D^\infty$ with $\frac{B^2}{16A} \in \mathbb{Z}[i]$ and $\frac{B^2}{16A} \equiv \pm 1(4)$. We make these assumptions as these are necessary conditions to study the sums of quartic exponential sums in this paper. Then the sum modulo Q of (10.6) equals

$$\sum_{d(Q)} \lambda(d) \theta(d) \left(\frac{d}{A}\right)_2 e\left(\frac{\bar{d}B^2}{4A}\right) = \sum_{d(Q)} \lambda(d) \theta(d) \left(\frac{d}{A}\right)_2.$$

As we are assuming $(D, 2) = 1$ by the Chinese remainder theorem this last sum equals

$$\left[\sum_{d|LCM(A,D)} \theta(d) \left(\frac{d}{A} \right)_2 \right] \left[\sum_{a(4)} \lambda(a) \right].$$

So for a non-zero contribution we must have $\lambda \equiv 1(4)$. We state when the d -sum is non-zero as a proposition. Define $T_{A,D}(\theta) := \sum_{d|LCM(A,D)} \theta(d) \left(\frac{d}{A} \right)_2$. We reduce to the case that $D = p \in \mathbb{Z}[i]$, p prime. So $A = p^j$, $j \geq 0$.

Proposition 10.1. *Let $T_{p^j,p}(\theta) := \sum_{d|p^j} \theta(d) \left(\frac{d}{p^j} \right)_2$. For $A = p^j$,*

$$(10.7) \quad T_{p^j,p}(\theta) = \begin{cases} \mathbb{N}(p)^j, & \text{if } j > 1, j \equiv 1(2) \text{ and } \theta = \left(\frac{\cdot}{p} \right)_2, \\ \mathbb{N}(p)^j, & \text{if } j > 1, j \equiv 0(2) \text{ and } \theta = 1, \\ \mathbb{N}(p), & \text{if } j = 1 \text{ and } \theta = \left(\frac{\cdot}{p} \right)_2, \\ \mathbb{N}(p), & \text{if } j = 0 \text{ and } \theta = 1, \\ 0, & \text{else.} \end{cases}$$

It is clear by multiplicativity,

$$T_{A,D}(\theta) = \prod_{p^j || LCM(A,D)} T_{p^j,p}(\theta).$$

As we assume A is a square, by the above Proposition, $T_{A,D}(\theta) = \mathbb{N}(A)$ if $\theta \equiv \mathbf{1}(D)$ and is zero if else.

Finally, we conclude for any $M \geq 0$ and X large,

$$(10.8) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4)}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) \sum_{x(c)} e\left(\text{Tr}\left(\frac{Ax^2 + Bx + F}{c}\right)\right) = \\ \frac{\sqrt{X} T_{A,D}(\theta)}{4} \int_{\mathbb{C}} \psi\left(\frac{1}{\mathbb{N}(t)}\right) e\left(\frac{B^2}{4A\sqrt{X}t}\right) e\left(\frac{F}{\sqrt{X}t}\right) \frac{dt}{\sqrt{\mathbb{N}(t)}} + O(X^{-M})$$

with

$$(10.9) \quad T_{A,D}(\theta) := \begin{cases} \mathbb{N}(A) & \text{if } \theta \equiv \mathbf{1}(D); \\ 0 & \text{if else.} \end{cases}$$

11. THE “CROSS” TERMS

11.1. Using Dirichlet characters to study the “cross” terms. The term to estimate for the “cross” terms is

$$(11.1) \quad \sum_{\substack{n, m \in \mathbb{Z}[i] \\ (n, m) = 1 \\ nm \equiv 1(4)}} \frac{\theta(nm) \psi_{X^{1/2-\delta}}(\mathbb{N}(n)) \psi_{X^{1/2+\delta}}(\mathbb{N}(m))}{\mathbb{N}(nm)} \left[\sum_{x(m)} e\left(\frac{\overline{n}(Ax^2 + Bx)}{m}\right) \right] \left[e\left(\frac{-B^2 \overline{8Am}}{n}\right) S_4(-B^2 \overline{m^4 A}, n) \right]$$

for each δ , $-1/2 < \delta < 1/2$.

Similar to the above case let us collect the m -sum together,

$$(11.2) \quad \sum_{n \in \mathbb{Z}[i]} \frac{\theta(n) \psi_{X^{1/2-\delta}}(\mathbb{N}(n))}{\mathbb{N}(n)} \sum_{x(n)^*} \rho(x) \left[\sum_{\substack{m \in \mathbb{Z}[i] \\ nm \equiv 1(4)}} \frac{\theta(m) \psi_{X^{1/2+\delta}}(\mathbb{N}(m))}{\sqrt{\mathbb{N}(m)}} e\left(\frac{\overline{m} \frac{P}{16}(x + \overline{x})}{n}\right) e\left(\frac{P \overline{m}}{n}\right) \right]$$

Using characters to detect the condition $nm \equiv 1(4)$, and writing

$$e\left(\frac{a}{n}\right) = \frac{1}{\phi(n)} \sum_{\chi(n)} \tau(\chi) \overline{\chi}(a),$$

the m -sum in (11.2) equals

$$\frac{\lambda(n)}{4\phi(n)^2} \sum_{\lambda(4)} \sum_{\chi_1, \chi_2(n)} \tau(\chi_1) \tau(\chi_2) \sum_{m \in \mathbb{Z}[i]} \frac{\psi_{X^{1/2+\delta}}(\mathbb{N}(m))}{\sqrt{\mathbb{N}(m)}} \lambda(m) \theta(m) \overline{\chi_1}\left(\overline{m} \frac{P}{4}(x + \overline{x})\right) \overline{\chi_2}(P \overline{m}).$$

Using Dirichlet characters here assumes $(x + \overline{x}, n) = 1$, we assume that for now. As the m -sum is finite, we write the sum over m as a sum of ideals,

$$(11.3) \quad \frac{\lambda(n)}{4\phi(n)^2} \sum_{\lambda(4)} \sum_{\chi_1, \chi_2(n)} \tau(\chi_1) \tau(\chi_2) \overline{\chi_1}\left(\frac{P}{4}(x + \overline{x})\right) \overline{\chi_2}(P) \sum_{\alpha \in \{\pm 1, \pm i\}} \overline{\chi_1 \chi_2}(\alpha) \theta(\alpha) \times \\ \sum_{(m)} \frac{\psi_{X^{1/2+\delta}}(\mathbb{N}(m))}{\sqrt{\mathbb{N}(m)}} \theta(m) \chi_1(m) \chi_2(m) \lambda(m).$$

By a standard Mellin inversion, the sum over ideals (m) equals for $\sigma > 3$,

$$\frac{1}{2\pi i} \int_{\Re(s)=\sigma} \tilde{\psi}(s) L(s + 1/2, \chi_1 \chi_2 \theta \lambda) X^{s(1/2+\delta)} ds$$

with $\tilde{\psi}$ the Mellin transform of ψ and

$$L(s, \chi_1 \chi_2 \theta \lambda) = \sum_{(m)} \frac{\chi_1(m) \chi_2(m) \theta(m) \lambda(m)}{\mathbb{N}(m)^s}.$$

If θ is a nontrivial character modulo D with $(D, n) = 1$, this Hecke L-function has analytic continuation to the entire complex plane with no poles. Therefore we can shift the contour to σ with $\Re(\sigma) = -M$ for any $M > 0$. This implies the integral is bounded by $O(X^{-M})$ for any $M > 0$. So if $(x + \overline{x}, n) = 1$, bounding the n -sum trivially then gives the bound for (11.2) as $O(X^{-M})$ for any $M > 0$.

Now let us assume that $(x + \bar{x}, n) > 1$. If $k = (x + \bar{x}, n) > 1$, then we can take as Dirichlet characters those of modulus $\frac{n}{k}$. The rest of the argument remains the same, and the Hecke L-function still has no poles. If $k = n$, then the term $e(\frac{\overline{m} \frac{P}{16}(x + \bar{x})}{n})$ does not exist and we only need χ_1 . Also in this case, as $(D, n) = 1$, the L-function has no pole and the bound $O(X^{-M})$ for any $M > 0$ holds generally for (11.2). We note also it did not matter what δ we chose in the range $(-1/2, 1/2)$.

11.2. $\theta \equiv \mathbf{1}(D)$. Now assume $\theta \equiv \mathbf{1}(D)$ and $\lambda \equiv \mathbf{1}(4)$, else the same above arguments work. Then we consider the analogous sum

$$(11.4) \quad \frac{1}{4\phi(n)^2} \sum_{\chi_1, \chi_2(n)} \tau(\chi_1)\tau(\chi_2)\overline{\chi_1}\left(\frac{P}{4}(x + \bar{x})\right)\overline{\chi_2}(P) \sum_{\alpha \in \{\pm 1, \pm i\}} \overline{\chi_1\chi_2}(\alpha)\theta(\alpha) \times \\ \sum_{(m)} \frac{\psi_{X^{1/2+\delta}}(\mathbb{N}(m))}{\sqrt{N(m)}} \chi_1(m)\chi_2(m).$$

The L-function $L(s + 1/2, \chi_1\chi_2)$ will now have a simple pole when $\chi_2 = \overline{\chi_1}$. By a contour shift to $\Re(s) = -M$ for $M > 0$, we can write (11.4) as

$$(11.5) \quad \frac{X^{1/4+\delta/2}\phi(4nD)\text{Res}_{s=1/2}\zeta_{\mathbb{Q}[i]}(s + 1/2)}{\mathbb{N}(4nD)\phi(n)^2} \sum_{\chi_1(n)} \tau(\chi_1)\tau(\overline{\chi_1})\overline{\chi_1}(\overline{4}(x + \bar{x})) + O(X^{-M}).$$

Using $\tau(\overline{\chi_1}) = \chi_1(-1)\overline{\tau(\chi_1)}$, we get

$$\frac{X^{1/4+\delta/2}\phi(4nD)\text{Res}_{s=1/2}\zeta_{\mathbb{Q}[i]}(s + 1/2)}{\mathbb{N}(4nD)\phi(n)} \sum_{\chi_1(n)} \chi_1(-\overline{4}(x + \bar{x})) + O(X^{-M}).$$

The sum in χ is nonzero when $x + \bar{x} \equiv -4(n)$, so incorporating the n -sum from (11.1) we have to estimate

$$(11.6) \quad \frac{X^{1/4+\delta/2}\text{Res}_{s=1/2}\zeta_{\mathbb{Q}[i]}(s + 1/2)\phi(4D)}{\mathbb{N}(4D)} \sum_{n \in \mathbb{Z}[i] - \{0\}} \frac{\psi_{X^{1/2-\delta}}(\mathbb{N}(n))}{\mathbb{N}(n)} \sum_{\substack{x(n)^* \\ x + \bar{x} \equiv -4(n)}} \rho(x).$$

So we conclude from this section that the “cross” terms are negligible, unless $\theta \equiv \mathbf{1}(D)$. In this case we have to understand (11.6) for each δ , $-1/2 < \delta < 1/2$. We certainly expect it to be much smaller than $X^{1/4}$ due to the oscillation of the term

$$\sum_{\substack{x(n)^* \\ x + \bar{x} \equiv -4(n)}} \rho(x).$$

Note if ρ is trivial then this term, by completing the square, is counting solutions to a quadratic equation modulo n . Such a sum is seen in counting geodesics on the modular surface in the work of Young and Soundararajan [SY]. Such a sum also arises in a subconvexity estimate for the Rankin-Selberg L-function in the author’s work [H].

12. SUMS OF QUARTIC EXPONENTIAL SUMS

Let $\psi \in C_0^\infty(\mathbb{R}^+)$ with $\psi_X(y) := \psi(\frac{X}{y})$ for $X > 0$ large. Let $D \in \mathbb{Z}[i]$, $D \equiv 1(4)$ and square free. Let θ be a Dirichlet character modulo D with $A, B, F \in \mathbb{Z}[i]$. Assume that $B = 4B'$ and $B' \parallel D^\infty$ with $\frac{B^2}{16A} \in \mathbb{Z}[i]$. Assume also that A, B are squares.

We investigate the asymptotic

$$(12.1) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) \sum_{x(c)} e(\text{Tr}(\frac{Ax^4 + Bx^2 + F}{c})).$$

Using Proposition 8.4, we can rewrite the sum (12.1) as

$$(12.2) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) e(\frac{F}{c}) \sum_{x(c)} e(\frac{Ax^4 + Bx^2}{c}) =$$

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) e(\frac{F}{c}) \sum_{x(c)} \chi(x) e(\frac{Ax^2 + Bx}{c}) + \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) e(\frac{F}{c}) \sum_{x(c)} e(\frac{Ax^2 + Bx}{c})$$

$$+ \{\text{cross terms}\} = \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) e(\frac{F}{c}) e(\frac{-B^2 \overline{8A}}{c}) S_\rho(\overline{4^2 AB^2}, \overline{4^2 AB^2}, c) +$$

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) e(\frac{F}{c}) \sum_{x(c)} e(\frac{Ax^2 + Bx}{c}) + \{\text{cross terms}\}.$$

12.1. First sum of (12.2). The first sum in (12.2) can be written using quadratic reciprocity as

$$(12.3) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c)) e(\frac{F}{c}) e(\frac{-B^2 \overline{8A}}{c})}{\mathbb{N}(c)} \theta(c) S_\rho(\overline{4^2 AB^2}, \overline{4^2 AB^2}, c).$$

Note as ψ is compactly supported on the positive reals, $\mathbb{N}(c) \gg X$, so for a fixed $G \in \mathbb{Z}[i]$,

$$\frac{c'G + cG'}{\mathbb{N}(c)} \ll \frac{\sqrt{\mathbb{N}(c)}}{\mathbb{N}(c)} \ll X^{-1/2}.$$

So

$$(12.4) \quad e(\frac{G}{c}) = \exp(2\pi i (\frac{c'G + cG'}{\mathbb{N}(c)})) = \sum_{j=0}^{\infty} \frac{(2\pi i \frac{c'G + cG'}{\mathbb{N}(c)})^j}{j!} = 1 + O(X^{-1/2}).$$

As $\frac{B^2}{8A} \in \mathbb{Z}[i]$ implies $\frac{-B^2}{8A} \equiv -B^2 \overline{8A}(c)$ for all $c \in \mathbb{Z}[i]$, we have by the above argument

$$e(\frac{F}{c}) e(\frac{-B^2 \overline{8A}}{c}) = e(\frac{F}{c}) e(\frac{\frac{-B^2}{8A}}{c}) = 1 + O(X^{-1/2}).$$

Therefore, we can write—using trivial bounds on the exponential sums—(12.3) as

$$(12.5) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) S_\rho(\frac{B^2}{16A}, \frac{B^2}{16A}, c) + O(X^\epsilon),$$

for any $\epsilon > 0$.

Using (9.6), for any $\epsilon > 0$ we have

$$(12.6) \quad \sum_{\substack{l \geq 1 \\ \mu_l \in (0,1]}} \overline{c_{\frac{B^2}{16A}, \sigma}(f_l)} c_{\frac{B^2}{16A}, \sigma'}(f_l) d_{\frac{B^2}{16A}, \sigma}(\mu_l) \overline{d_{\frac{B^2}{16A}, \sigma'}(\mu_l)} \frac{X^{\mu_l} \tilde{\psi}(\mu_l)}{g(-\mu_l)} =$$

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) S_\rho\left(\frac{B^2}{16A}, \frac{B^2}{16A}, c\right) + O(X^\epsilon).$$

With Theorem 7.1, we know there exists $\mu_l \in (0, 1]$, namely $\mu_l = \frac{1}{4}$ if $\theta^4 = 1$ from a residual Eisenstein series f_{00} , and that there is no other term $\mu_l \in (0, 1]$ with $\mu_l > \frac{7}{256}$. To keep clear of what family of metaplectic forms we are using, the residual Eisenstein series (as well as all metaplectic forms used in this trace formula) associated to this sum of quartic sums, transforms on left by Γ by $\theta\kappa$.

So, for any $\epsilon > \frac{7}{256}$,

$$(12.7) \quad \frac{X^{\frac{1}{4}} \tilde{\psi}(\frac{1}{4})}{g(-\frac{1}{4})} \overline{c_{\frac{B^2}{16A}, \sigma}(f_{00})} c_{\frac{B^2}{16A}, \sigma'}(f_{00}) d_{\frac{B^2}{16A}, \sigma}\left(\frac{1}{4}\right) \overline{d_{\frac{B^2}{16A}, \sigma'}\left(\frac{1}{4}\right)} =$$

$$\sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) S_\rho\left(\frac{B^2}{16A}, \frac{B^2}{16A}, c\right) + O(X^\epsilon).$$

12.2. Second sum of (12.2). For the second sum we use (10.8),

$$(12.8) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c,D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) \sum_{x(c)} e\left(\text{Tr}\left(\frac{Ax^2 + Bx + F}{c}\right)\right) =$$

$$\frac{\sqrt{X} T_{A,D}(\theta)}{2} \int_{\mathbb{C}} \psi\left(\frac{1}{\mathbb{N}(t)}\right) e\left(\frac{B^2}{4A\sqrt{X}t}\right) e\left(\frac{F}{\sqrt{X}t}\right) \frac{dt}{\sqrt{\mathbb{N}(t)}} + O(X^{-M})$$

where

$$(12.9) \quad T_{A,D}(\theta) := \begin{cases} \mathbb{N}(A) & \text{if } \theta \equiv \mathbf{1}(D); \\ 0 & \text{if else.} \end{cases}$$

12.3. Third sum of (12.2). This term is what we studied in Section 11. There we concluded these “cross” terms are of size $O(X^{-1/2})$, at the largest, or if $\theta \equiv \mathbf{1}(D)$ there exists a term, for each $\delta \in (-1/2, 1/2)$ which we cannot immediately evaluate:

$$(12.10) \quad \frac{X^{1/4+\delta/2} \text{Res}_{s=1/2} \zeta_{\mathbb{Q}[i]}(s+1/2) \phi(4D)}{\mathbb{N}(4D)} \sum_{n \in \mathbb{Z}[i] - \{0\}} \frac{\psi_{X^{1/2-\delta}}(\mathbb{N}(n))}{\mathbb{N}(n)} \sum_{\substack{x(n)^* \\ x+\bar{x} \equiv -4(n)}} \rho(x).$$

Let us write it as

$$R(\theta) \frac{\text{Res}_{s=1/2} \zeta_{\mathbb{Q}[i]}(s+1/2) \phi(4D)}{\mathbb{N}(4D)} \int_{(-1/2, 1/2)} X^{1/4+\delta/2} \left[\sum_{n \in \mathbb{Z}[i] - \{0\}} \frac{\psi_{X^{1/2-\delta}}(\mathbb{N}(n))}{\mathbb{N}(n)} \sum_{\substack{x(n)^* \\ x+\bar{x} \equiv -4(n)}} \rho(x) \right] d\delta,$$

$$\text{with } R_D(\theta) := \begin{cases} 1 & \text{if } \theta \equiv \mathbf{1}(D) \\ 0 & \text{if else.} \end{cases}$$

12.4. Conclusion. After the analysis of these three terms, we conclude for any $\epsilon > \frac{7}{256}$,

$$(12.11) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c, D) = 1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) \sum_{x(c)} e(\text{Tr}(\frac{Ax^4 + Bx^2 + F}{c})) =$$

$$\frac{X^{\frac{1}{2}} T_{A,D}(\theta)}{2} \int_{\mathbb{C}} \psi(\frac{1}{\mathbb{N}(t)}) e(\frac{B^2}{4A\sqrt{X}t}) e(\frac{F}{\sqrt{X}t}) \frac{dt}{\sqrt{\mathbb{N}(t)}} +$$

$$\frac{X^{\frac{1}{4}} \tilde{\psi}(\frac{1}{4})}{g(-\frac{1}{4})} c_{\frac{B^2}{16A}, \sigma}(f_{00}) c_{\frac{B^2}{16A}, \sigma'}(f_{00}) d_{\frac{B^2}{16A}, \sigma}(\frac{1}{4}) d_{\frac{B^2}{16A}, \sigma'}(\frac{1}{4}) +$$

$$R_D(\theta) \frac{\text{Res}_{s=1/2} \zeta_{\mathbb{Q}[i]}(s+1/2) \phi(4D)}{\mathbb{N}(4D)} \int_{(-1/2, 1/2)} X^{1/4+\delta/2} \left[\sum_{n \in \mathbb{Z}[i] - \{0\}} \frac{\psi_{X^{1/2-\delta}}(\mathbb{N}(n))}{\mathbb{N}(n)} \sum_{\substack{x(n)^* \\ x+\bar{x} \equiv -4(n)}} \rho(x) \right] d\delta$$

$$+ O(X^\epsilon).$$

Let $\hat{\psi}(s)$ denote the Mellin transform over \mathbb{R} ,

$$\hat{\psi}(s) = \int_0^\infty \psi(y) y^{s-1} dy.$$

To make the result more symmetric in ψ , we can write using the same argument as in (12.4) to get

$$\int_{\mathbb{C}} \psi(\frac{1}{\mathbb{N}(t)}) e(\frac{B^2}{4A\sqrt{X}t}) e(\frac{F}{\sqrt{X}t}) \frac{dt}{\sqrt{\mathbb{N}(t)}} = \int_{\mathbb{C}} \psi(\frac{1}{\mathbb{N}(t)}) \frac{dt}{\sqrt{\mathbb{N}(t)}} + O(X^{-1/2}).$$

Then using a change of variables to polar coordinates

$$\int_{\mathbb{C}} \psi(\frac{1}{\mathbb{N}(t)}) \frac{dt}{\sqrt{\mathbb{N}(t)}} = 2\pi \int_0^\infty \psi(\frac{1}{y^2}) dy = \pi \hat{\psi}(-\frac{1}{2}).$$

Likewise,

$$\tilde{\psi}(\frac{1}{4}) = \pi \hat{\psi}(-\frac{1}{4}) \text{ and } \tilde{\psi}(0) = \pi \hat{\psi}(0).$$

The error from using the Taylor expansion is of size

$$X^{1/2} \cdot O(X^{-1/2}) = O(1)$$

and we contain it in the error $O(X^\epsilon)$.

We write

$$\frac{d_{\frac{B^2}{16A}, \sigma}(\frac{1}{4}) \overline{d_{\frac{B^2}{16A}, \sigma'}(\frac{1}{4})}}{g(-\frac{1}{4})} = \frac{\Gamma(3/4)^2 \mathbb{N}(\frac{B^2}{16A})^{1/4}}{(-4\pi)^2} \cdot \frac{4}{(8\pi)^{1/4} \Gamma(5/4)^2 \mathbb{N}(\frac{B^2}{16A})^{1/4}} = \frac{-\Gamma(3/4)^2}{\pi^2 (8\pi)^{1/4} \Gamma(5/4)^2}.$$

The class number formula tells us

$$\text{Res}_{s=1/2} \zeta_{\mathbb{Q}[i]}(s+1/2) = L(1, \chi_{\mathbb{Q}[i]}),$$

with $\chi_{\mathbb{Q}[i]}$ the quadratic character associated to $\mathbb{Q}[i]$.

Putting all this together, we get (12.11) equals

$$(12.12) \quad \sum_{\substack{c \in \mathbb{Z}[i] \\ c \equiv 1(4) \\ (c, D)=1}} \frac{\psi_X(\mathbb{N}(c))}{\mathbb{N}(c)} \theta(c) \sum_{x(c)} e(\text{Tr}(\frac{Ax^4 + Bx^2 + F}{c})) =$$

$$X^{\frac{1}{2}} \pi \hat{\psi}(-\frac{1}{2}) T_{A,D}(\theta) +$$

$$X^{\frac{1}{4}} \pi \hat{\psi}(-\frac{1}{4}) \frac{-\Gamma(3/4)^2}{\pi^2 (8\pi)^{1/4} \Gamma(5/4)^2} \frac{|c_{\frac{B^2}{16A}, \sigma}(f_{00})|^2}{\text{vol}(\Gamma_{N^\sigma}/\Gamma)} +$$

$$R_D(\theta) \frac{L(1, \chi_{\mathbb{Q}[i]}) \phi(4D)}{\mathbb{N}(4D)} \int_{(-1/2, 1/2)} X^{1/4+\delta/2} \left[\sum_{n \in \mathbb{Z}[i] - \{0\}} \frac{\psi_{X^{1/2-\delta}}(\mathbb{N}(n))}{\mathbb{N}(n)} \sum_{\substack{x(n)^* \\ x+\bar{x} \equiv -4(n)}} \rho(x) \right] d\delta + O(X^\epsilon).$$

In proving (12.12), we complete our theorem.

13. APPENDIX: BRUGGEMAN-MIATELLO KUZNETSOV TRACE FORMULA OVER AN IMAGINARY QUADRATIC FIELD WITH MULTIPLE CUSPS

Definition 13.1. For $\psi > 0$ and $a > 3$ define $\mathcal{H}_a(\psi)$ as the set of even holomorphic functions h on $|\Re \nu| \leq 2\psi$ satisfying

$$h(\nu) \ll e^{\frac{-|\Im \nu|}{2}} (1 + |\Im \nu|)^{-a}.$$

Definition 13.2. For $r \in \mathbb{Z}[i] \setminus \{0\}$, and $h \in \mathcal{H}_a(\psi)$, define the function $K_r h$ on G by

$$K_r h(g) := \sqrt{\mathbb{N}(r)} \frac{i}{2} \int_Y h(\nu) W_{\nu, r}(g) \sin \pi \nu d\nu$$

with $W_{\nu, r}(na[y]k) := \chi_r(n) \sqrt{\mathbb{N}(y)} K_\nu(4\pi y \sqrt{\mathbb{N}(r)})$.

Fix non-equivalent essential cusps σ, σ' . We have the Poincare series

$$P_{\sigma, \psi}^r(g) = \sum_{\Gamma_{N^\sigma} \backslash \Gamma} (\theta(\gamma) \kappa(\gamma))^{-1} \chi_r(n(\gamma g)) \psi(a(\gamma g)),$$

with $\psi(a) := K_r h(a)$. Note it transforms by $P_{\sigma, \psi}^r(\gamma g) = \theta(\gamma) \kappa(\gamma) P_{\sigma, \psi}^r(g)$. The absolute convergence of the series, and in particular the following integrals of this variant of the Bruggeman-Miatello trace formula follow immediately from the analysis of [[BM], Section 10], and we will not mention such issues anymore.

Take a nontrivial square integrable automorphic $f_l(g)$, then following directly the procedure for taking the inner product of a Poincare series with an automorphic form in [[BMP1],5.3] we have

$$\langle P_{\sigma,\psi}^r, f_l \rangle = \frac{\pi^2}{2} \overline{c_{r,\sigma}(f_l)} d_{r,\sigma}(\mu_l) \sqrt{\mathbb{N}(r)} h(\mu_l).$$

We recall from [[BM],[BMP1]] that $c_{r,\sigma}(f_l)$ is r -th Fourier coefficient of the automorphic form f_l at the cusp σ and

$$d_{r,\sigma}(\nu) = \frac{1}{\text{vol}(\Gamma_{N^\sigma} \setminus \Gamma)} \frac{2^{1-\nu} (4\pi \sqrt{N(r)})^{-\nu}}{\Gamma(\nu+1)}.$$

Remark. There seems to be a discrepancy between the factor $d_{r,\sigma}(\nu)$ in [BM] and [BMP1] for totally real fields. But checking with [IK] and [L], it should be as we defined it for an imaginary quadratic field.

13.1. Spectral computation of the scalar product. Using the standard spectral decomposition of an inner product of automorphic forms $\langle f, f_1 \rangle$, with $\psi' := K_r h'$ we have

$$\langle P_{\sigma,\psi}^r, P_{\sigma',\psi'}^{r'} \rangle = \frac{\pi^4}{4} \sqrt{\mathbb{N}(rr')} \int_Y k(\nu) d\zeta_{r,r',\sigma,\sigma'}(\nu).$$

Here for an even test function η

$$(13.1) \quad \int_Y \eta(\nu) d\zeta_{r,r',\sigma,\sigma'}(\nu) = \sum_{l \geq 1} \overline{c_{r,\sigma}(f_l)} c_{r',\sigma'}(f_l) d_{r,\sigma}(\mu_l) \overline{d_{r',\sigma'}(\mu_l)} \eta(\mu_l) + \\ \sum_{\beta \in \mathcal{P}} \sum_{\mu} \int_{-\infty}^{\infty} \overline{D_{r,\sigma}(P^\beta, iyp, i\mu)} D_{r',\sigma'}(P^\beta, iyp, i\mu) d_{r,\sigma}(\mu_l) \overline{d_{r',\sigma'}(\mu_l)} \eta(iyp + i\mu) dy.$$

13.2. Geometric computation of the scalar product. Following [[BMP1],58] we can expand the scalar product as

$$\langle P_{\sigma,\psi}^r, P_{\sigma',\psi'}^{r'} \rangle = \sum_{\gamma \in \Gamma_{N^{\sigma'}} \setminus \Gamma} I(\gamma),$$

with

$$I(\gamma) = (\theta(\gamma) \kappa(\gamma))^{-1} \int_A a^{-2\rho} \psi(g_\sigma a) \int_{\Gamma_{N^\sigma}} \chi_r(n) \psi'(\gamma n g_\sigma a) dnd a.$$

We write this as $I_1 + I_2$,

$$I_1 := \sum_{\Gamma_{N^{\sigma'}} \setminus (\Gamma \cap g_{\sigma'} P g_{\sigma'}^{-1})} I(\gamma),$$

and

$$I_2 := \sum_{\gamma \in \Gamma_{N^{\sigma'}} \setminus {}^{\sigma'} \Gamma^\sigma} I(\gamma).$$

We remind that ${}^{\sigma'} \Gamma^\sigma = \Gamma \cap g_{\sigma'} C g_{\sigma'}^{-1}$ with $C = (P \cap G_{\mathbb{Q}}) s_0 (N \cap G_{\mathbb{Q}})$ and $s_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Define the measure

$$\int_Y f(\nu) d\eta(\nu) := \frac{i}{2} \int_{\Re(\nu)=0} f(\nu) \sin \pi \nu d\nu.$$

For I_1 we write just as in [BM],

$$I_1 = \frac{\pi^2}{4} \sqrt{\mathbb{N}(rr')} \Delta_{r,r'}^{\sigma,\sigma'}(k) = \frac{\pi^2}{4} \sqrt{\mathbb{N}(rr')} \text{vol}(\Gamma_{N^\sigma} \setminus N^\sigma) \alpha(\kappa, r, \kappa', r') \int_Y k(\nu) d\eta(\nu)$$

with $k(x) = h(x)\overline{h'}(x)$ and $\alpha(\kappa, r, \kappa', r') = \sum_{\gamma} (\theta(\gamma)\kappa(\gamma))^{-1} \chi_{r'}(n_\sigma(\gamma g_\sigma)^{-1})$ with the sum over $\gamma \in \Gamma_{N^{\sigma'}} \setminus (\Gamma \cap g_{\sigma'} P g_\sigma^{-1})$ for which $\chi_r(n) = \chi_{r'}(\gamma n \gamma^{-1})$ for all $n \in N^\sigma$. This is similar to the Definition 2.6.1 in [BMP1].

Using the exact same reasoning as [BMP1],

$$I_2 = \sum_{\sigma' \in \mathcal{L}^\sigma} \sum_{\delta \in \Gamma_{N^\sigma}} I(\gamma\delta).$$

Let $c \in {}^{\sigma'}\mathcal{C}^\sigma, \gamma \in {}^{\sigma'}\mathcal{L}^\sigma(c)$ following their Definition 2.3.1 and Proposition 2.3.2. Write $\xi = g_{\sigma'}^{-1} \gamma g_\sigma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix} = n[\xi] m[\xi] a_\xi s_0 n'[\xi] \in SL_2(\mathbb{Q}(i))$, with

$$(13.2) \quad n[\xi] m[\xi] a_\xi s_0 n'[\xi] = \begin{pmatrix} 1 & \frac{a(\gamma)}{c(\gamma)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c(\gamma)} & 0 \\ 0 & c(\gamma) \end{pmatrix} s_0 \begin{pmatrix} 1 & \frac{d(\gamma)}{c(\gamma)} \\ 0 & 1 \end{pmatrix}.$$

Writing $c = c(\gamma)$ for economy, we have

$$(13.3) \quad \sum_{\delta \in \Gamma_{N^\sigma}} I(\gamma\delta) = (\theta(\gamma)\kappa(\gamma))^{-1} \int_A a^{-2\rho} K_r h(g_\sigma a) \int_{N^\sigma} \chi_r(n) \overline{K_{r'} h'(\gamma n g_\sigma a)} dnd a = \\ \pi \sqrt{\mathbb{N}(r'/r)} (\theta(\gamma)\kappa(\gamma))^{-1} \overline{\chi_{r'}(n'(\xi)) \chi_r(n(\xi))} \int_A a^{-2\rho} K_r h(g_\sigma a) K_{r'} h'_{1/c^2}(g_\sigma a) da$$

with $h_t(\nu) := \sqrt{N(t)} h(\nu) \mathbf{B}(rr't^2, \nu)$ and

$$\mathbf{B}(t, \nu) := \frac{I_\nu(4\pi\sqrt{t}) I_\nu(4\pi\sqrt{t}) - I_{-\nu}(4\pi\sqrt{t}) I_{-\nu}(4\pi\sqrt{t})}{\sin \pi \nu}.$$

Again, this notation is analogous to [BM] as we are working over an imaginary quadratic field.

Following (83) of [BMP1],

$$\int_A a^{-2\rho} K_r h(g_\sigma a) K_{r'} h'_{1/c^2}(g_\sigma a) da = 2\pi^3 \mathbb{N}(r) \int_Y h(\nu) h'_{1/c^2}(\nu) d\eta(\nu).$$

Opening the definition of h_{1/c^2} and writing $k = h\overline{h'}$,

$$(13.4) \quad \sum_{\delta \in \Gamma_{N^\sigma}} I(\gamma\delta) = \frac{\pi^3}{2} \sqrt{\mathbb{N}(r'r)} (\theta(\gamma)\kappa(\gamma))^{-1} \overline{\chi_{r'}(n'(\xi)) \chi_r(n(\xi))} |\mathbb{N}(c)|^{-1} \int_Y k(\nu) \mathbf{B}\left(\frac{rr'}{c^2}, \nu\right) d\eta(\nu) = \\ \frac{\pi^3}{2} \sqrt{\mathbb{N}(r'r)} (\theta(\gamma)\kappa(\gamma))^{-1} \overline{\chi_{r'}(n'(\xi)) \chi_r(n(\xi))} |\mathbb{N}(c)|^{-1} Bk\left(\frac{rr'}{c^2}\right).$$

The last equality uses Definition 5.7 from [BM].

Now recalling our dependence of a, b, c, d on γ , the expression (13.2),

$$\overline{\chi_{r'}(n'(\xi)) \chi_r(n(\xi))} = e(\text{Tr}_{\mathbb{Q}(i)/\mathbb{Q}}(\frac{r'd(\gamma)}{c(\gamma)} + \frac{ra(\gamma)}{c(\gamma)})).$$

Let

$$\sigma' \Gamma^\sigma(c) = \{\gamma \in \sigma' \Gamma^\sigma : g_{\sigma'}^{-1} \gamma g_\sigma = \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix}\},$$

then Proposition 2.3.2 from [BMP1] has an analogous argument for an imaginary quadratic field which can be expressed for a test function f as

$$(13.5) \quad \sum_{\gamma \in \sigma' \mathcal{L}^\sigma} f(\gamma) = \sum_{c \in \sigma' \mathcal{C}^\sigma} \sum_{\gamma \in \Gamma_{N^{\sigma'}} \setminus \sigma' \Gamma(c)^\sigma / \Gamma_{N^\sigma}} f(\gamma).$$

Label $\sigma' \mathcal{L}(c)^\sigma := \Gamma_{N^{\sigma'}} \setminus \sigma' \Gamma(c)^\sigma / \Gamma_{N^\sigma}$. Using (13.4) and (13.5),

$$(13.6) \quad I_2 = \sum_{\sigma' \mathcal{L}^\sigma} \sum_{\delta \in \Gamma_{N^\sigma}} I(\gamma \delta) = \frac{\pi^3}{2} \sum_{c \in \sigma' \mathcal{C}^\sigma} S_{\kappa \theta}^{\sigma, \sigma'}(r, r', c) \sqrt{\mathbb{N}(r' r)} |\mathbb{N}(c)|^{-1} Bk\left(\frac{rr'}{c^2}\right)$$

where we have defined for a multiplicative character ψ on $SL_2(\mathbb{Z}[i])$,

$$S_{\psi}^{\sigma, \sigma'}(r, r', c) := \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in g_{\sigma'}^{-1} \sigma' \mathcal{L}(c)^\sigma g_\sigma} \left(\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)^{-1} e(-Tr_{\mathbb{Q}(i)/\mathbb{Q}}(\frac{r'd}{c} + \frac{ra}{c})).$$

So we have expressed the inner product of two Poincare series in two different ways giving

$$(13.7) \quad \frac{\pi^4}{4} \sqrt{\mathbb{N}(rr')} \left[\sum_{l \geq 1} \overline{c_{r, \sigma}(f_l)} c_{r', \sigma'}(f_l) d_{r, \sigma}(\mu_l) \overline{d_{r', \sigma'}(\mu_l)} k(\mu_l) + \right. \\ \left. \sum_{j=1}^m \sum_{\mu} \int_{-\infty}^{\infty} \overline{D_{r, \sigma}(P^j, i y \rho, i \mu)} D_{r', \sigma'}(P^j, i y \rho, i \mu) d_{r, \sigma}(\mu_l) \overline{d_{r', \sigma'}(\mu_l)} k(i y \rho + i \mu) dy \right] = \\ \frac{\pi^2 i}{8} \sqrt{\mathbb{N}(rr')} \text{vol}(\Gamma_{N^\sigma} \setminus N^\sigma) \alpha(\sigma, r, \sigma', r') \times \\ \int_{\Re(\nu)=0} k(\nu) \sin \pi \nu d\nu + \frac{\pi^3}{2} \sqrt{\mathbb{N}(r' r)} \sum_{c \in \sigma' \mathcal{C}^\sigma} S_{\kappa \theta}^{\sigma, \sigma'}(r, r', c) |\mathbb{N}(c)|^{-1} Bk\left(\frac{rr'}{c^2}\right).$$

Dividing by $\frac{\pi^4}{4} \sqrt{\mathbb{N}(rr')}$ we get

$$(13.8) \quad \sum_{l \geq 1} \overline{c_{r, \sigma}(f_l)} c_{r', \sigma'}(f_l) d_{r, \sigma}(\mu_l) \overline{d_{r', \sigma'}(\mu_l)} k(\mu_l) + \\ \sum_{j=1}^m \sum_{\mu} \int_{-\infty}^{\infty} \overline{D_{r, \sigma}(P^j, i y \rho, i \mu)} D_{r', \sigma'}(P^j, i y \rho, i \mu) d_{r, \sigma}(\mu_l) \overline{d_{r', \sigma'}(\mu_l)} k(i y \rho + i \mu) dy \Big] = \\ \frac{i}{2} \text{vol}(\Gamma_{N^\sigma} \setminus N^\sigma) \alpha(\kappa, r, \kappa', r') \int_{\Re(\nu)=0} k(\nu) \sin \pi \nu d\nu + \frac{\pi}{2} \sum_{c \in \sigma' \mathcal{C}^\sigma} S_{\kappa \theta}^{\sigma, \sigma'}(r, r', c) |\mathbb{N}(c)|^{-1} Bk\left(\frac{rr'}{c^2}\right).$$

This trace formula holds with the choices of $h, h' \in \mathcal{H}_a(\psi)$ and $k = h\overline{h'}$. Like [BM], we give such k a definition:

Definition 13.3. For $\psi > 0$ and $a > 6$ define $\mathcal{K}_a(\psi)$ as the set of even holomorphic functions k on $|\Re \nu| \leq 2\psi$ satisfying

$$k(\nu) \ll e^{-|\Im \nu|} (1 + |\Im \nu|)^{-a}.$$

Compare this to Theorem 2.7.1 of [BMP1] and Theorem 6.1 of [BM].

14. APPENDIX 2: EXPONENTIAL SUM IDENTITIES

We look at the case of Theorem 8.1 for a higher power.

Proposition 14.1. *Let $p, A, B \in \mathbb{Z}[\mu_{2n}]$, with $p \equiv 1(n)$, $(AB, p) = 1$. Denote the character of order 2 as η . Then, we have the following identity*

$$\sum_{x(p)} e\left(\frac{Ax^{2n} + Bx^n}{p}\right) = \sum_{\substack{\rho(n) \\ \xi^2=\rho}} \frac{\rho(B^2)\tau(\eta)\eta(-1)}{\mathbb{N}(p)} \sum_{a,b(p)} \xi(a)\eta(a)\xi(b)e\left(\frac{a+b+4ab\overline{B^2}A}{p}\right).$$

Proof.

$$(14.1) \quad \sum_{A(p)} \overline{\chi(A)} \sum_{x(p)} \rho(x) e\left(\frac{Ax^2 + Bx}{p}\right).$$

By a change of variables $A \rightarrow A\overline{x^2}$, $x \rightarrow \overline{B}x$ this equals

$$\chi(\overline{B^2})\rho(B^2)\tau(\chi^2\rho)\tau(\overline{\chi}).$$

So by Fourier inversion

$$(14.2) \quad \sum_{x(p)} \rho(x) e\left(\frac{Ax^2 + Bx}{p}\right) = \rho(B^2) \frac{1}{\phi(p)} \sum_{\chi(p)} \chi(\overline{B^2}) \tau(\chi^2\rho) \tau(\overline{\chi}) \chi(A).$$

The Hasse Davenport relation gives the identity for $\chi(p)$,

$$(14.3) \quad \tau(\chi^n) = \frac{-\chi(n^n) \prod_{l(n)} \tau(\chi\gamma^l)}{\prod_{l(n)} \tau(\gamma^l)},$$

where γ is the n -th residue character.

Suppose $\rho = \xi^2$, one can consider ξ a $2n$ -th residue character.

Writing the term $\tau(\chi^2\rho) = \tau((\chi\xi)^2) = \tau((\chi\xi)^2)$, and using the Hasse-Davenport relation we can write

$$\tau((\chi\xi)^2) = \frac{\chi(4)\tau(\chi\xi\eta)\tau(\chi\xi)\tau(\eta)\eta(-1)}{\mathbb{N}(p)}.$$

This equality can be reached by using the equality $\overline{\tau(\eta)} = \eta(-1)\tau(\eta)$.

So (14.2) equals

$$\frac{\rho(B^2)\tau(\eta)\eta(-1)}{\mathbb{N}(p)\phi(p)} \sum_{\chi(p)} \chi(\overline{B^2})\chi(A) [\chi(4)\tau(\chi\xi\eta)\tau(\chi\xi)\tau(\overline{\chi})].$$

Opening all the Gauss sums and rearranging sums this equals

$$\frac{\eta(B)\tau(\eta)\eta(-1)}{\mathbb{N}(p)\phi(p)} \sum_{a(p)} \xi(a)\eta(a)e\left(\frac{a}{p}\right) \sum_{b(p)} \xi(b)e\left(\frac{b}{p}\right) \sum_{c(p)} e\left(\frac{c}{p}\right) \sum_{\chi(p)} \chi(4abc\overline{B^2}A).$$

Using orthogonality of characters this equals

$$\frac{\rho(B^2)\tau(\eta)\eta(-1)}{\mathbb{N}(p)} \sum_{\substack{a,b,c(p) \\ 4abA \equiv B^2c(p)}} \xi(a)\eta(a)e\left(\frac{a}{p}\right)\xi(b)e\left(\frac{b}{p}\right)e\left(\frac{c}{p}\right).$$

We can rewrite this as

$$\frac{\rho(B^2)\tau(\eta)\eta(-1)}{\mathbb{N}(p)} \sum_{a,b(p)} \xi(a)\eta(a)\xi(b)e\left(\frac{a+b+4ab\overline{B^2}A}{p}\right).$$

But unlike Theorem 8.1, this cannot be reduced to a rank 2 Kloosterman sum. This sum is a close relation to a hypergeometric sum found in [K]. So the power $n = 2$ is a special case that the exponential sums in question are related to rank 2 Kloosterman sums. \square

If one had to guess what exponential sums are associated to rank 2 Kloosterman sums, one could look at the prime power modulus, say $p^m, m > 1$ for p in some number field. From Proposition 8.2, it is necessary that the polynomial of the exponential sum looks like $f(x) = Ax^n + Bx^{n-2} + C$ for A, B, C integers in some number field. Indeed, Proposition 8.2 tells us that the derivative needs to satisfy some relation $F(A, B, C)x^2 \equiv G(A, B, C)(p^j)$, for $1 \leq j \leq m$, with $F(A, B, C), G(A, B, C)$ some functions of A, B, C . One can imitate the same method done for Proposition 14.1 above for such an exponential sum

$$\sum_{x(p)} e\left(\frac{Ax^n + Bx^{n-2} + C}{p}\right),$$

and realize that is not related, at least via an analogous Hasse-Davenport argument, to a Kloosterman sum. Even for $n = 4$, it is not clear what to do, until one applies characters of order 2 to the sum and then one arrives at Theorem 8.1. All of this argument is not a proof that no other exponential sums of higher degree polynomials are related to rank 2 Kloosterman sums, but is good evidence the degree 4 sums we study are likely the highest power related to rank 2 Kloosterman sums and therefore for metaplectic forms of any cover of GL_2 .

It is clear these sums can be related to hyper-Kloosterman sums or higher rank Kloosterman or Hypergeometric sheafs of [K], but an important question is to link a sum of such sums or sheafs to something spectral like automorphic forms via the trace formula. This seems to be a difficult question as it is still not clear whether rank 3 Kloosterman sheafs are related to GL_3 automorphic forms or some subset (automorphic representations of $U(3)$?) of them.

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